Numer. Math. Theor. Meth. Appl. Vol. **16**, No. 1, pp. 230-241
doi: 10.4208/nmtma.OA-2022-0082 February 2023

Constructing Order Two Superconvergent WG Finite Elements on Rectangular Meshes

Xiu Ye[1,*] and Shangyou Zhang[2]

[1] *Department of Mathematics, University of Arkansas at Little Rock, Little Rock, AR 72204, USA*
[2] *Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA*

Received 13 May 2022; Accepted (in revised version) 22 July 2022

Abstract. In this paper, we introduce a stabilizer free weak Galerkin (SFWG) finite element method for second order elliptic problems on rectangular meshes. With a special weak Gradient space, an order two superconvergence for the SFWG finite element solution is obtained, in both L^2 and H^1 norms. A local post-process lifts such a P_k weak Galerkin solution to an optimal order P_{k+2} solution. The numerical results confirm the theory.

AMS subject classifications: 65N15, 65N30

Key words: Finite element, weak Galerkin method, stabilizer free, rectangular mesh.

1. Introduction

A new stabilizer free weak Galerkin method is developed to solve the following second order elliptic problem:

$$-\Delta u = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = g \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$, which can be subdivided into rectangular meshes.

The weak Galerkin (WG) finite element methods introduced in [24, 25] provide a general finite element technique for solving partial differential equations. The novelty of the WG method is the introduction of weak function and its weakly defined derivatives. The weak functions possess the form of $v = \{v_0, v_b\}$ with $v = v_0$ representing the value of v in the interior of each element and $v = v_b$ on the boundary of the element.

*Corresponding author. *Email addresses:* `xxye@ualr.edu` (X. Ye), `szhang@udel.edu` (S. Zhang)

The weak derivative $\nabla_w v$ for a weak function v is defined as distributions. WG method uses polynomials $(P_k(T), P_s(e), [P_\ell(T)]^d)$ to approximate $(v_0, v_b, \nabla_w v)$ accordingly. The WG methods have been applied for solving various PDEs such as Sobolev equation, the Navier-Stokes equations, the Oseen equations, time-dependent Maxwell's equations, elliptic interface problems, biharmonic equations, etc, [1, 5–17, 21–23, 26, 27, 30].

For some special combinations of the WG element $(P_k(T), P_s(e), [P_\ell(T)]^d)$, stabilizer is no longer needed in the corresponding weak Galerkin finite element formulations, which leads to a stabilizer free weak Galerkin method. The stabilizer free weak Galerkin method was first introduced in [28] on polygonal/polyhedral meshes and then has been applied for the second order problems, the Stokes equations and the biharmonic equation [2, 18, 29].

This paper has two purposes:

1. Developing a new SFWG method with an order two superconvergence for the problem (1.1)-(1.2).

2. Providing necessary theory for a subsequent paper, order two superconvergent conforming discontinuous Galerkin method on rectangular meshes.

A WG element $(P_k(T),\ P_{k+1}(e), \mathrm{BDM}_{[k]}[T])$ on rectangular mesh is used in this stabilizer free weak Galerkin finite element method. We prove that the SFWG method converges to the true solution of (1.1)-(1.2) with a convergence rate two orders higher than the optimal order in both an energy norm and the L^2 norm theoretically and numerically. We further define a local post-process which lifts such a P_k weak Galerkin solution to an optimal order P_{k+2} solution. It is proved and numerically verified.

2. The weak Galerkin finite element scheme

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of rectangles. Denote by $\mathcal{E}_h$ the set of all edges in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ be the set of all interior edges. For every element $T \in \mathcal{T}_h$, we denote by h_T its diameter and the mesh size by $h = \max_{T\in\mathcal{T}_h} h_T$ for $\mathcal{T}_h$.

For a given integer $k \geq 1$, let V_h be the weak Galerkin finite element space associated with $\mathcal{T}_h$ defined as follows:

$$V_h = \big\{v = \{v_0, v_b\} : v_0|_T \in P_k(T), v_b|_e \in P_{k+1}(e), e \subset \partial T, T \in \mathcal{T}_h\big\} \tag{2.1}$$

and its subspace V_h^0 is defined as

$$V_h^0 = \{v : v \in V_h, v_b = 0 \ \text{ on } \ \partial\Omega\}. \tag{2.2}$$

We would like to emphasize that any function $v \in V_h$ has a single value v_b on each edge $e \in \mathcal{E}_h$.

On each rectangle $T \in \mathcal{T}_h$, the BDM finite element space is defined by [4]

$$\mathrm{BDM}_{[k+1]}(T) = P_{k+1}(T)^2 \oplus \mathbf{curl} x^{k+2} y \oplus \mathbf{curl} x y^{k+2}.$$