INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 20, Number 2, Pages 153–175

doi: 10.4208/ijnam2023-1007

LINEAR MOMENT MODELS TO APPROXIMATE KNUDSEN LAYERS

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Abstract. We propose a well-posed Maxwell-type boundary condition for the linear moment system in half-space. As a reduction model of the Boltzmann equation, the moment equations are available to model Knudsen layers near a solid wall, where proper boundary conditions play a key role. In this paper, we will collect the moment system into the form of a general boundary value problem in half-space. Utilizing an orthogonal decomposition, we separate the part with a damping term from the system and then impose a new class of Maxwell-type boundary conditions on it. Due to the block structure of boundary conditions, we show that the half-space boundary value problem admits a unique solution with explicit expressions. Instantly, the well-posedness of the linear moment system is achieved. We apply the procedure to classical flow problems with the Shakhov collision term, such as the velocity slip and temperature jump problems. The model can capture Knudsen layers with very high accuracy using only a few moments.

Key words. Knudsen layer, half-space moment system, Maxwell-type boundary condition, well-posedness.

1. Introduction

The Knudsen layer is an important rarefaction effect of gas flows near the surface [25], where non-Maxwellian velocity distribution functions must be considered because of the gas-surface interaction. The gas exhibits non-Newtonian behavior in the Knudsen layer, and there is a finite velocity or temperature gap at the surface, known as the velocity slip or temperature jump [21, 37]. A better understanding of the Knudsen layer may help design numerical methods for coupling the Boltzmann and Euler equations [13, 39], avoiding solving the complex multidimensional Boltzmann equation in the whole space.

The linearized Boltzmann equation (LBE) [38] is widely used to depict the Knudsen layer. Half-space problems for the Boltzmann equation are often solved by the direct simulation Monte Carlo (DSMC) method [7] or discrete velocity/ordinates method (DVM/DOM) [8, 32, 5]. Numerical results for various collision models have been reported [26, 27, 32]. In theory, the well-posedness has been exhaustively studied for linear half-space kinetic equations [3, 23, 24] and discrete Boltzmann equations in the DVM [5].

Meanwhile, Grad's moment method [16] has been developed [35, 9, 10] into a popular reduction model of the Boltzmann equation with efficient numerical methods [11, 28, 20]. It is also available [34, 17, 14] to model the Knudsen layer. Compared with kinetic equations, the moment system often gives a formal analytical general solution and leads to empirical formulas describing the gas behavior in the Knudsen layer. These formulas may help simplify the coupling of the Knudsen layer and bulk solutions. However, the Maxwell boundary condition proposed by Grad [16] is shown unstable [30] even in the linearized case. For the linear initial-boundary

Received by the editors on April 12, 2022; accepted on November 2, 2022.
2000 *Mathematics Subject Classification.* 34B40, 34B05, 35Q35, 76P05, 82B40.

value problem (IBVP) of moment equations, [30] has defined stability criteria and constructed the formulation of stable boundary conditions.

To our best knowledge, the well-posedness results are still scattered for half-space problems based on Grad's arbitrary order moment equations with Maxwell-type boundary conditions. In numeric, there are also few universal methods to deal with different flow problems. For example, [14] numerically solves Kramers' problem for the BGK [6] model and proves the well-posedness when the accommodation coefficient is an algebraic number. This paper aims to overcome these two lacks.

One of our main contributions is to propose a new class of Maxwell-type boundary conditions. It makes sure the well-posedness of the linear homogeneous moment system in half-space. The system is derived from Grad's moment equations under some Knudsen layer assumptions and can deal with different specific flow problems with various collision terms. We first collect the system into a general half-space boundary value problem. Then we make an orthogonal decomposition to separate the equations with a damping term from the whole system. With the method of characteristics, we get several well-posedness criteria about boundary conditions. Under these criteria, the solvability of the moment system is ensured. From the constructive proof, we can even write explicit analytical solutions to the moment system. The procedure gives an efficient algorithm to solve half-space problems, which is another contribution of this paper.

Specifically, the construction of boundary conditions mainly follows Grad's idea [16, 11] by imposing the continuity of odd fluxes [2] at the boundary. The obtained Maxwell-type boundary conditions are in a common even-odd parity form [34, 14]. To meet the well-posedness criteria for half-space problems, the linear space determined by Maxwell-type boundary conditions is encouraged to contain the null space of the boundary matrix. This idea has been emphasized in many other problems [22, 29, 30, 40]. We will first get redundant boundary conditions and then combine them linearly to meet the above criteria.

This paper is organized as follows. In Section 2, we summarize the main well-posedness result of linear half-space boundary value problems. In Section 3, we derive the moment system in half-space with Maxwell-type boundary conditions. In Section 4, we apply our model to velocity slip and temperature jump problems with the Shakhov collision term. The paper ends with a conclusion.

2. Solvability Conditions for Half-Space Problems

We consider the boundary layer problem arising in rarefied gas flows when the Knudsen number tends to zero. These equations are often linear [1] regardless of the nonlinear setting of the original equations. Meanwhile, the resulting half-space moment system may have block structures due to the orthogonality and the recursion relation of Hermite polynomials.

Therefore, we consider the linear half-space boundary value problem with constant coefficients

$$\begin{aligned} \boldsymbol{A}\frac{\mathrm{d}\boldsymbol{w}(y)}{\mathrm{d}y} &= -\boldsymbol{Q}\boldsymbol{w}(y), \quad y \in [0, +\infty), \\ \boldsymbol{w}(+\infty) &= \boldsymbol{0}, \end{aligned} \tag{1}$$

where $\boldsymbol{w}(y) \in \mathbb{R}^{m+n}$ with $m \geq n$, and

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{M} \\ \boldsymbol{M}^T & \boldsymbol{0} \end{bmatrix}, \qquad \boldsymbol{Q} = \begin{bmatrix} \boldsymbol{Q}_e & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q}_o \end{bmatrix}. \tag{2}$$