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Blow-Up Phenomena for Some Pseudo-Parabolic Equations with Nonlocal Term

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Abstract. We investigate the initial boundary value problem of some semilinear pseudo-parabolic equations with Newtonian nonlocal term. We establish a lower bound for the blow-up time if blow-up does occur. Also both the upper bound for T and blow up rate of the solution are given when $J(u_0) < 0$. Moreover, we establish the blow up result for arbitrary initial energy and the upper bound for T. As a product, we refine the lifespan when $J(u_0) < 0$.

Key Words: Nonlocal pseudo-parabolic equations, blow-up, upper bound, lower bound.

AMS Subject Classifications: 35B44, 35K70

1 Introduction

In this paper, we are concerned with the following initial boundary value problem (IBVP) of some pseudo-parabolic equations with nonlocal term

$$\begin{cases} u_t - \Delta u_t - \Delta u + u = a\phi_u u + b|u|^{p-1}u, & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,t) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $p \in (1,5)$ and ϕ_u is the Newtonian nonlocal term

$$\phi_u(x) = \int_\Omega \frac{u^2(y)}{4\pi|x-y|} dy, \quad x \in \mathbb{R}^3. \tag{1.2}$$

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Elliptic problems with the nonlocal term $\phi_u u$ have been extensively investigated in the field of physic and mathematics by many authors, for example, the Klein-Gordon-Maxwell equations [1,2] and the Schrödinger-Poisson-Slater problem [3]. Recently, the local and global existence of solutions for the parabolic equations with the nonlocal term $\phi_u u$ were considered in [4]. More specifically, they considered the following three cases: (A) $a = b = 1$; (B) $a = -1, b = 1$; (C) $a = 1, b = 0$.

Since the appearance of $-\Delta u_t$, Eq. (1.1) is called the pseudo-parabolic equation, which describes many interesting physical and biological phenomena, for example, the non-stationary process in semiconductors in the presence of sources and $\Delta u_t - u_t$ stands for the free electron density rate (see [5]). Pseudo-equations have been extensively studied by many authors since the work of Ting [6,7], see for example [8–14] and references therein. Most of these papers were interested in the existence and the blow up of the solutions to the pseudo-equations with polynomial nonlinear source term (i.e., $a = 0$, $b = 1$). Recently, the existence and blow up of solutions for the pseudo-equation with nonlocal term $\phi_u u$ have been investigated, that is the IBVP(1.1) with Case (A) $a = b = 1$ [15] and the IBVP (1.1) with Case (B) $a = -1, b = 1$ [16]. The existence, asymptotic behavior and blow up results were obtained respectively in [15,16] via energy equality together with the potential wells methods. By the way, the following nonlocal problem

$$u_t - \Delta u = \Big(\int_\Omega \frac{|u(y)|^p}{|x-y|^{n-2}} dy \Big) |u|^{p-2} u, \quad x \in \Omega \subset R^n, \quad t > 0,$$

has been investigated in [17–20].

In this paper, we will continue to study the blow up properties of the IBVP (1.1). For convenience, we denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and the associated inner product on $L^2(\Omega)$. Moreover, we use $\|\cdot\|_p$ and $\|\cdot\|_{H_0^1}$ to denote the norms on $L^p(\Omega)$ and $H_0^1(\Omega)$) respectively. We also denote C as the different constant in different line and T as the maximal existence time.

First, we introduce the definition of the weak solution to the IBVP (1.1).

Definition 1.1 ([15,16]). *A function $u \in L^2_{loc}([0,T), H_0^1(\Omega))$ with $u_t \in L^2_{loc}([0,T), H_0^1(\Omega))$ is called a weak solution of IBVP* (1.1) *provided*

(i) for a.e. $t \in [0,T)$ it holds that

$$\begin{aligned} &(u_t(t), v) + (\nabla u_t(t), \nabla v) + (u(t), v) + (\nabla u(t), \nabla v) \\ =&(a\phi_u u + b|u|^{p-1}u, v), \quad v \in H_0^1(\Omega). \end{aligned}$$

(ii) $u(x,0) = u_0(x)$.

In order to compare with our work in this paper, For the completeness, we summarize the existence and blow-up results obtained in [15,16] as following.