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ADAPTIVE CENTRAL-UPWIND SCHEME ON TRIANGULAR GRIDS FOR THE SHALLOW WATER MODEL WITH VARIABLE DENSITY

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Abstract. In this paper, we construct a robust adaptive central-upwind scheme on unstructured triangular grids for two-dimensional shallow water equations with variable density. The method is well-balanced, positivity-preserving, and oscillation free at the curve where two types of fluid merge. The proposed approach is an extension of the adaptive well-balanced, positivity-preserving scheme developed in Epshteyn and Nguyen (arXiv preprint arXiv:2011.06143, 2020). In particular, to preserve "lake-at-rest" steady states, we utilize the Riemann Solver with appropriately rotated coordinates to obtain the point values in neighborhood of the fluid interface. In addition, to improve the efficiency of an adaptive method in the multi-fluid flow, the curve of density discontinuity is reconstructed by using the level set method and volume fraction method. To demonstrate the accuracy, high-resolution, and efficiency of the new adaptive central-upwind scheme, several challenging tests for Shallow water models with variable density are performed.

Key words. Shallow water equations with variable density, central-upwind scheme, well-balanced and positivity-preserving scheme, adaptive algorithm, interface tracking, Riemann solver, weak local residual error estimator, unstructured triangular grid.

1. Introduction

The main goal of this paper is to develop an adaptive well-balanced positivity-preserving central-upwind scheme on triangular grids for shallow water equations with variable density (SWEDs). The two-dimensional (2-D) system of SWEDs can be written as,

$$w_t + (hu)_x + (hv)_y = 0, \tag{1a}$$

$$(hu)_t + \Big(hu^2 + \frac{g}{2\rho_0}h^2\rho\Big)_x + (huv)_y = -\frac{g}{\rho_0}h\rho B_x, \tag{1b}$$

$$(hv)_t + (huv)_x + \Big(hv^2 + \frac{g}{2\rho_0}h^2\rho\Big)_y = -\frac{g}{\rho_0}h\rho B_y, \tag{1c}$$

$$(h\rho)_t + (hu\rho)_x + (hv\rho)_y = 0, \tag{1d}$$

where t is the time, x and y are spatial coordinates ($(x, y) \in \Omega$), $h(x, y, t)$ is the water height, $B(x, y)$ is the bottom topography, $w(x, y, t) := h + B$ is the water level, $\rho(x, y, t)$ is the density, $u(x, y, t)$ and $v(x, y, t)$ are the x- and y-components of the flow velocity, g is the constant gravitational acceleration, and ρ_0 is the reference density. The system (1a)–(1d) was proposed in [10, 36, 37, 15] as a variation of the Saint-Venant equations to model multi-phase flows in estuaries or deep ocean currents. The derivation of the system is based on hydrostatic approximation which eliminates the variability in the z-direction. The design of robust and accurate

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numerical algorithms for computing the solutions of SWEDs system is an important and challenging problem that has been extensively studied in the recent years.

A number of numerical schemes for balance laws have been introduced in recent years, [28, 25, 21, 6, 27, 8, 7, 9, 24, 23, 3, 26, 4, 5, 40, 29]. Most of them utilize a Riemann problem solver for the upwind evolution of the calculated solution. However, as discussed in [9], the eigensystem of the system (1a)–(1d) may be incomplete due to the resonance phenomenon. Hence, it may be very difficult to design a reliable upwind scheme for the SWEDs. In our paper, we therefore use central-upwind schemes which are Riemann-problem-solver free methods, [22, 25, 29]. Central-upwind schemes have been referred to "black-box" solvers for general multidimensional systems of hyperbolic systems of conservation laws. In our prior work [13], we have derived a successful adaptive central-upwind method for Saint-Venant system on triangular grids. We then adapt the developed adaptive scheme in [13] to the new system (1a)–(1d).

Similar to the Saint-Venant system, a good method for SWEDs system should preserve the non-negativity of h and ρ, which is called the positivity-preserving property. In addition, the scheme must ensure a well-balanced property obtained when the numerical method preserve "lake-at-rest" steady-state solutions. Otherwise, the numerical method may lead to significant oscillations. Note that, the system (1a)–(1d) admits the following two "lake-at-rest" steady-state solutions, [9]:

$$w := h + B = \max\big\{C, B(x,y)\big\}, \quad C = \text{Const}, \quad \rho = \text{P} \equiv \text{Const}, \quad \text{u} \equiv \text{v} \equiv 0, \tag{2}$$

and

$$B \equiv \text{Const}, \quad \text{h}^2\rho \equiv \text{Const}, \quad \text{u} \equiv \text{v} \equiv 0. \tag{3}$$

Preserving the solution (3) is a big challenge for numerically solving the system (1a)–(1d) since using the conventional central-upwind methods may not ensure the variable $h^2\rho$, so-called variable pressure, to be constant at the contact waves. It is more difficult to prevent the density oscillation when working on the unstructured triangular grids. Several numerical methods have been proposed for compressible flows, see [46, 31, 9], but only a few efforts, see [9], can simultaneously ensure two types of lake-at-rest states. Therefore, we consider the approach in [9] which is derived to solve the shallow water model with horizontal temperature gradients on rectangular meshes (the system in [9] has similar properties with the SWEDs). In [9], the proposed second-order semi-discrete central-upwind scheme is capable of preserving the ąřlake at restąś steady state (2) and (3) as well as the positivity of the water depth and the temperature (the variable temperature is equivalent to the variable density in our work). In particular, to preserve the second type of ąřlake at restąś steady state (3) and suppress the pressure oscillations across the interface, an efficient interface tracking method is performed. The main idea of the interface approach in [9] is to completely avoid to use the information from the cells where two types of fluids are numerically mixed, so-called "mixed" cells when evolving the solution in the neighborhood of the interface. The data in the ąřmixedąś cells is replaced by the interpolated values that are calculated using the reliable information from the nearby ąřsingle fluidąś cells. Namely, the point values in "mixed" cells are obtained by using the approximated solution of the 1-D Riemann problems between the reliable ąřsingle fluidąś cell averages. However, the central-upwind method and the interface tracking in [9] are designed for structured rectangular grids. In practice, one needs to deal with complicated geometries, where the use of triangular grids could be advantageous or even unavoidable. Hence, in this study, we extend the interface tracking method [9] from the rectangular grids