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Boundedness of the Multilinear Maximal Operator with the Hausdorff Content

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Abstract. In this paper, we establish the strong and weak boundedness of the multilinear maximal operator in the setting of the Choquet integral with respect to the α-dimensional Hausdorff content. Our results cover Orobitg and Verdera's results in [8].

Key Words: Multilinear maximal operator, Hausdorff content, Choquet integrals.

AMS Subject Classifications: 42B25, 42B35

1 Introduction

The purpose of this paper is to establish the strong and weak boundedness of the multilinear maximal operator on the Choquet space. For m-couple locally integrable functions $(f_1,\cdots,f_m)$ on $\mathbb{R}^n\times\cdots\times\mathbb{R}^n$, the multi(sub)linear maximal operator M is defined by

$$M(f_1,\cdots,f_m)(x):=\sup_{Q\ni x}\prod_{i=1}^{m}\frac{1}{|Q|}\int_Q|f_i(y)|\,\mathrm{d}y, \tag{1.1}$$

where the supremum is taken over all cubes Q containing x with sides parallel to the coordinate axes. Very often it is much more convenient to work with dyadic multilinear maximal function $M_d(f_1,\cdots,f_m)$, which is defined by the right-hand side of (1.1), but the supremum is taken only on the family of dyadic cubes containing x. Clearly, when $m=1$, M is the classical Hardy-Littlewood maximal operator. These maximal operators are fundamental tools to study harmonic analysis, potential theory, and the theory of partial differential equations (see, e.g., [3,5]).

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For $E \subset \mathbb{R}^n$ and $0 < \alpha \leq n$, the α-dimensional Hausdorff content of E is defined by

$$H^{\alpha}(E) := \inf \sum_{j=1}^{\infty} \ell(Q_j)^{\alpha}, \tag{1.2}$$

where the infimum is taken over all coverings of E by countable families of cubes Q_j with sides parallel to the coordinate axes and $\ell(Q)$ denotes the side length of the cube Q. If we take the infimum in (1.2) only on coverings of E by dyadic squares, we can obtain an equivalent quantity $H_d^{\alpha}(E)$ called the dyadic α-dimensional Hausdorff content. In [8], Orobitg and Verdera used the Choquet integral with respect to the α-dimensional Hausdorff content to extend some well-known estimates for Hardy-Littlewood maximal opertaor. They proved the strong type inequality

$$\int (Mf)^p \, \mathrm{d}H^{\alpha} \leq C \int |f|^p \, \mathrm{d}H^{\alpha} \tag{1.3}$$

for $\alpha/n < p$, and the weak type inequality

$$H^{\alpha}\{x\colon Mf(x) > t\} \leq Ct^{-\frac{\alpha}{n}} \int |f|^{\frac{\alpha}{n}} \, \mathrm{d}H^{\alpha} \tag{1.4}$$

for any $t > 0$ and $p = \alpha/n$. Here, the integrals are taken in the Choquet sense, that is, the Choquet integral of $\varphi \geq 0$ with respect to a set function Λ is defined by

$$\int \varphi \, \mathrm{d}\Lambda := \int_0^{\infty} \Lambda\{x \in \mathbb{R}^n\colon \varphi(x) > t\} \, \mathrm{d}t.$$

When $\alpha = n$, both (1.3) and (1.4) become the classical strong type inequality and weak type inequality, respectively. It is worth mentioning that the Orobitg-Verdera result came from their efforts to comprehend the special case $p = 1$ that is first proved by Adams in [1]–a result of the H^1-BMO duality theory applied to the characterization of the Riesz capacities. In fact, the Orobitg-Verdera's proof is a modification of arguments due to Carleson [4] and Hormander [6]. Moreover, Tang [10] generalized the preceding results and established the boundedness of maximal operators on the weighted Choquet space and the Choquet-Morrey space.

Motivated by these works, we investigate the strong and weak boundedness of the multilinear maximal operators in the frame of Choquet integrals with respect to the α-dimensional Hausdorff content.

Now, we formulate our main results as follows.

Theorem 1.1. *Let $0 < \alpha < n$, $0 < p \leq p_i < \infty$ with $1 \leq i \leq m$ such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\frac{\alpha}{n} < \min\{p_1, \cdots, p_m\}$. Then, the following inequality*

$$\left(\int \big(M(f_1, \cdots, f_m)\big)^p \, \mathrm{d}H^{\alpha} \right)^{\frac{1}{p}} \leq C \prod_{i=1}^{m} \left(\int |f_i|^{p_i} \, \mathrm{d}H^{\alpha} \right)^{\frac{1}{p_i}}$$

holds for some constant C depending on α, m, n and p_i.