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Numerical Study on Viscous Fingering Using Electric Fields in a Hele-Shaw Cell

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Abstract. We investigate the nonlinear dynamics of a moving interface in a Hele-Shaw cell subject to an in-plane applied electric field. We develop a spectrally accurate numerical method for solving a coupled integral equation system. Although the stiffness due to the high order spatial derivatives can be removed using a small scale decomposition technique, the long-time simulation is still expensive since the evolving velocity of the interface drops dramatically as the interface expands. We remove this physically imposed stiffness by employing a rescaling scheme, which accelerates the slow dynamics and reduces the computational cost. Our nonlinear results reveal that positive currents restrain finger ramification and promote the overall stabilization of patterns. On the other hand, negative currents make the interface more unstable and lead to the formation of thin tail structures connecting the fingers and a small inner region. When no fluid is injected, and a negative current is utilized, the interface tends to approach the origin and break up into several drops. We investigate the temporal evolution of the smallest distance between the interface and the origin and find that it obeys an algebraic law $(t_*-t)^b$, where t_* is the estimated pinch-off time.

AMS subject classifications: 45B05, 35R37, 76D27, 76S05, 76W05

Key words: Hele-Shaw problem, fingering instabilities, electro-osmotic flow, boundary integral method, rescaling idea.

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1 Introduction

Interfacial instabilities are ubiquitous in nature and engineering, such as dendritic growth in solidification [8, 23, 32, 47, 53], fractal growth of diffusion-limited aggregation [52, 61], electrodeposition of metals [3, 9], various patterns of tissue [1, 15, 37, 48, 57], viscous fingering in Hele-Shaw cells [13, 17, 35, 42, 43, 51, 56, 62], and so on. In many cases, these instabilities are not desired. For instance, viscous fingering results in trapping of oil in the reservoir, thus leading to poor oil recovery [24, 55]; dendritic growth leads to safety problems in rechargeable batteries [63]; vascular tumor induces difficulty in clinic treatment [12, 39]. The interfacial instability has been an attractive topic for decades.

Viscous fingering in the Hele-Shaw cell, where a small gap separates two parallel plates, can be used as a prototype for investigation of interfacial instabilities [27]. When a less viscous fluid is injected into the cell and displaces a more viscous fluid, the interface separating the two fluids experiences the well-known Saffman-Taylor instabilities and as a consequence, viscous fingering patterns are formed [16, 33, 51]. As the interface expands, these fingers split at their tips, generating new fingers. Finally, the interface performs dense branching morphologies [7, 13, 35, 40, 43, 46].

Recently, one variant of the classical Hele-Shaw set-up that catches researchers' attention is the Hele-Shaw problem coupled with an applied electric field [20, 41]. In this set-up, an external electric field is utilized to promote electro-osmotic flow, which is added to the pressure-driven flow. Electric-field-induced dynamics have also been explored in other circumstances, such as in the study of the three-dimensional motion of drops and vesicles [30, 60] subjected to electric fields. In the context of electro-osmotic flows in Hele-Shaw cells, Mirzadeh and Bazant [41] carried out a linear stability analysis of the problem in a Hele-Shaw channel, and found that the interfacial instabilities could be fully suppressed by employing sufficiently large currents. In [20], Gao *et al.* analyzed the same problem but in a radial Hele-Shaw displacement, where the electro-osmotic flow can either oppose or assist the pressure-driven flow. By carrying out various experiments, they also found that the electric current is able to actively control the emergence of interfacial instabilities. Although the authors provided a very clear physical explanation for the observed phenomenon, their results are limited (by their power source) to small values of electric current, which do not permit a systematic analysis of the effects of the electric field on radial viscous fingering. In addition, the experiments performed in [20] are focused on the onset of the pattern formation, where the size of the fingers is still small and nonlinear effects are not significant. To our best knowledge, the nonlinear mechanisms that dictate the growth of instabilities are still not fully understood, especially for limiting morphologies at long times. Motivated by these facts, our goal in the current work is to develop an efficient and accurate numerical scheme to investigate the long-time evolution of the interface during injection-driven, electro-osmotic, radial Hele-Shaw flow.

To better understand the interfacial dynamics, we develop a boundary integral formulation to simulate the system efficiently and accurately. Herein the hydraulic and