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Weak Galerkin Method for Second-Order Elliptic Equations with Newton Boundary Condition

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Abstract. The weak Galerkin (WG) method is a nonconforming numerical method for solving partial differential equations. In this paper, we introduce the WG method for elliptic equations with Newton boundary condition in bounded domains. The Newton boundary condition is a nonlinear boundary condition arising from science and engineering applications. We prove the well-posedness of the WG scheme by the monotone operator theory and the embedding inequality of weak finite element functions. The error estimates are derived. Numerical experiments are presented to verify the theoretical analysis.

AMS subject classifications: 65N30, 65N15, 35J65

Key words: Weak Galerkin method, Newton boundary condition, monotone operator, embedding theorem.

1 Introduction

In this paper, we consider the following second-order elliptic equations with Newton boundary condition:

$$-\Delta u = f, \quad \text{in } \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial \mathbf{n}} + \kappa |u|^{\alpha} u = g, \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a polygonal domain in $\mathbb{R}^2$, f and g are given functions, and $\kappa > 0$, $\alpha \geqslant 0$ are constants. Such boundary value problems are widely used in science and engineering, such as aluminum electrolytic modeling problems [19], radiation heat transfer problems [18] and nonlinear elasticity problems [11,12].

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In the past two decades, a great deal of efforts have been devoted to this problem. In [7], Feistauer and Najzar apply the finite element method to solve the problem in two-dimensional polygonal domains. After that, Feistauer et al. derived error estimates of the finite element method in [10]. They pointed out that the H^1 error does not achieve optimal convergence order and the convergence of H^1 norm is related to the parameter in the boundary condition. Eqs. (1.1)-(1.2) in the non-polygonal domain are considered in [8], and the corresponding error estimates are given in [23]. Furthermore, in [1], O. Bartoš et al. prove that the nonlinearity only slows down the convergence of L^2 norm when the weak solution is zero on the entire boundary. Feistauer et al. investigate this problem by using the DG method in [6, 9], and they conclude that the error estimates depend essentially on the opening angle of the corner points and the nonlinearity in the boundary term. As pointed out in [13, 17], the weak Galerkin method is closely related to other numerical methods such as the hybridizable discontinuous Galerkin method [5] and the mixed finite element method. For ease of expression, this paper uses the framework of the weak Galerkin method to construct the algorithm. The idea of proof is also adoptable for the hybridizable discontinuous Galerkin method.

The goal of this paper is to investigate the weak Galerkin method for (1.1)-(1.2). The WG method is first proposed for the second-order elliptic equations by Wang and Ye in [27], and further developed in [14, 20–22, 25, 27, 28, 30, 38]. The main idea of the weak Galerkin method is to use discontinuous piecewise polynomials as basis functions, and to replace the classical derivative operators by specifically defined weak derivative operators in the numerical scheme. In the past few years, the WG methods have been applied to various partial differential equations, such as the Stokes equations [24,26,29,31,32,37], the Brinkman problem [33, 36], the biharmonic equations [39], eigenvalue problems [35], integro-differential equations [34], stochastic problems [15, 40] and so on.

The main challenges in solving partial differential equations with Newton boundary condition are establishing the existence and uniqueness of the numerical solution and the convergence of errors. In order to solve these problems, we use the monotone operator theory and the embedding inequality for the weak functions to prove the well-posedness of the WG scheme. Furthermore, with the help of the embedding inequality and some other techniques, we prove that the convergence order is related to the parameter of the nonlinear term and whether the exact solution is zero on the whole boundary.

This paper is organized as follows. In Section 2, we introduce some notations, definitions, and the WG scheme. In Section 3, we prove the embedding inequality for the weak functions and the well-posedness of the WG scheme. The error estimates for the WG approximations are given in Section 4. Some numerical experiments are presented in Section 5.

2 The weak Galerkin scheme

In this section, we introduce the WG scheme for (1.1)-(1.2). Throughout the paper, we follow the usual notations for Sobolev spaces $W^{m,p}(\Omega)$ and norms $\|\cdot\|_{W^{m,p}(\Omega)}$.