INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 20, Number 3, Pages 437–458

doi: 10.4208/ijnam2023-1018

A FULLY IMPLICIT METHOD USING NODAL RADIAL BASIS FUNCTIONS TO SOLVE THE LINEAR ADVECTION EQUATION

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Abstract. Radial basis functions are typically used when discretization schemes require inhomogeneous node distributions. While spawning from a desire to interpolate functions on a random set of nodes, they have found successful applications in solving many types of differential equations. However, the weights of the interpolated solution, used in the linear superposition of basis functions to interpolate the solution, and the actual value of the solution are completely different. In fact, these weights mix the value of the solution with the geometrical location of the nodes used to discretize the equation. In this paper, we used nodal radial basis functions, which are interpolants of the impulse function at each node inside the domain. This transformation allows to solve a linear hyperbolic partial differential equation using series expansion rather than the explicit computation of a matrix inverse. This transformation effectively yields an implicit solver which only requires the multiplication of vectors with matrices. Because the solver requires neither matrix inverse nor matrix-matrix products, this approach is numerically more stable and reduces the error by at least two orders of magnitude, compared to solvers using radial basis functions directly. Further, boundary conditions are integrated directly inside the solver, at no extra cost. The method is locally conservative, keeping the error virtually constant throughout the computation.

Key words. Radial basis functions, implicit scheme, hyperbolic equations.

1. Introduction

Radial basis function interpolation is one of the few methods that can approximate across a d-dimensional space a function only defined on a randomly distributed set of n nodes $x_1, x_2, ..., x_n \in \mathbb{R}^d$ [17, 14]. While initially used for interpolation problems, this method can be used to define surfaces in multiple dimensions [3] or solve partial differential equations [18, 19, 6, 9]. One important characteristic of radial basis functions is their definition using the relative position of nodes, obtained from the Euclidian norm $\| \cdot \|_d$, rather than their absolute location in space. For $x \in \mathbb{R}^d$ we define the radial basis function (RBF) at every node x_j as $\Phi_{x_j}(x) = \phi(||x - x_j||_d)$, which we will also write as $\Phi(x - x_j)$. Typically, Φ is normalized, i.e. $\Phi(0) = 1$. New functions can be generated by scaling of the modal function ϕ by a factor α, giving the standard definition of the radial basis function Φ_{α,x_j} as

$$\Phi_{\alpha,x_j}(x) = \phi(||x - x_j||_d/\alpha),$$

which we will also write as $\Phi_\alpha(x - x_j)$. We use the width parameter α rather than the usual shape factor (i.e. $1/\alpha$) in this paper because we will compare the radial basis function spread to the domain size throughout this paper.

A continuous function f can be approximated on a finite set of nodes $U = \{x_1, x_2, ..., x_n\}$ using radial basis functions by computing a set of weights ω_j defined by

$$\forall x_i \in U, f\left(x_i\right) = \sum_j \omega_j \Phi_\alpha\left(x_i - x_j\right).$$

Received by the editors on February 9, 2022 and, accepted on January 26, 2023.
2000 *Mathematics Subject Classification.* 65D12, 65F05, 35F50, 35F46, 65D05, 35L65.

The weights ω_j can be found by solving the linear system

$$\begin{bmatrix} \Phi_\alpha(x_1 - x_1) & ... & \Phi_\alpha(x_1 - x_n) \\ ... & ... & ... \\ \Phi_\alpha(x_n - x_1) & ... & \Phi_\alpha(x_n - x_n) \end{bmatrix} \begin{bmatrix} \omega_1 \\ ... \\ \omega_n \end{bmatrix} = \begin{bmatrix} f(x_1) \\ ... \\ f(x_n) \end{bmatrix}$$

written in compact form as $[\Phi_\alpha][\omega] = [f]$. To solve this system, we need to find the inverse of the matrix $[\Phi_\alpha]$ and compute the weights ω_j using

$$[\omega] = [\Phi_\alpha]^{-1}[f]. \tag{1}$$

The radial basis function Φ is said to be definite positive when $[\Phi]$ is invertible, supposing U does not have any redundant nodes (i.e. $x_i{=}x_j$ while $i \neq j$). Once the weights are known, the function f can be interpolated between nodes using the function $\overline{f}$ defined by

$$\overline{f}(x) = \sum_j \omega_j \Phi_\alpha \left(x - x_j\right). \tag{2}$$

Since radial basis functions can interpolate any smooth function using a linear combination of differentiable functions, it quickly spawned differential equation solvers for elliptic [24], hyperbolic [23], parabolic [36] or shallow-water [11] equations using different approaches such as spectral [30] or backward substitution [27, 37] methods. Even differential equations with fractional operators [25, 22], curvilinear coordinates [33] or complex boundary conditions [20] can be solved using this technique. The ease in defining spacial and temporal derivatives is probably the main reason this method has found universal applications.

However, one major issue raised by Eq. (2) is evident. The interpolated function $\overline{f}$ is now defined in term of the weights ω_j. This becomes an issue when solving differential equations using radial basis functions. For instance, the value of the function might be required to compute the value of another function or match a set of boundary conditions. Further if we want to interpolate a new function g, then all the weights ω_j must be computed again.

In the rest of the paper, we first define a set of nodal radial basis functions (N-RBF) that interpolates the impulse function. These functions form an orthonormal basis for the inner product of an interpolated function on U. First, we summarize the basic properties of NRBF formed using RBF with compact support, then we present the theory behind our linear advection equation solver and compare it to standard solvers. Finally, we conclude by showing how NRBF can be trivially extended to solve the advection equation with a velocity which varies across the domain. It is important to note that the method is completely independent of the number of spatial dimensions by construction. As a result, we will not look at multidimensional cases in this paper. While we do not claim that the method will perform well in a larger number of dimensions, the solver proposed is clearly dimension agnostic.

2. Definition of nodal radial basis functions

The solution to avoid computing f from the w_j is relatively straightforward. Rather than using radial basis functions directly, which have well defined, yet poorly matched, values at the node points, we can used them to interpolate the impulse functions $\delta(x - x_i)$ first. Once these new functions Ψ_{x_i} are defined, interpolation is trivial since the weights for each interpolant Ψ_{x_i} is $f(x_i)$. To construct them, we