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Operators on Sobolev Type Spaces

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Abstract. In this paper, we introduce the work done on Hardy-Sobolev spaces and Fock-Sobolev spaces and their operators and operator algebras, including the study of boundedness, compactness, Fredholm property, index theory, spectrum and essential spectrum, norm and essential norm, Schatten-p classes, and the C^* algebras generated by them.

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Key words: Hardy-Sobolev space, Fock-Sobolev space, multiplier, composition operator, Toeplitz operator, Hankel operator.

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1 Introduction

The operator theory on analytical function space, which has nearly one hundred years of history, is studied as an important branch in the field of complex analysis and functional analysis. It is closely related to control theory, probability theory, quantum mechanics, partial differential equations, harmonic analysis, and computational mathematics, has been one of the important research directions of modern mathematics. In all analytic function spaces, the Hardy space has formed the most abundant theory. For example, Beurling's theorem characterizes the invariant subspace of the coordinate multiplier T_z on the Hardy space of the unit circle; the internal and external factorization well describes the structure of analytic functions in this kind of space (see [41]).

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The real Sobolev type space is the product of the combination of the classical Sobolev space structure and the classical function space of real variables (such as Hardy space, BMO space, Orlitz space, etc.). It originates from the estimation of the boundedness of the singular integral operator with non-smooth kernel in harmonic analysis. Due to the lack of analytical structure, the study of this type of space and its operator algebras is complicated, and the essential difficulty lies in the estimation of relevant inequalities. If it is given an analytical structure, such space may be studied by using analytic function theory.

Hardy-Sobolev spaces and Fock-Sobolev spaces, as two new kinds of analytic Sobolev type spaces, contain almost all the classical analytic function spaces, not only broader than any classical analytic space, but also more special than the classical Sobolev space. What is the spatial structure of these two types of analytical type Sobolev spaces? What are the structure and properties of the multiplicative operators, Toeplitz operators and composition operators on these spaces? We in this paper introduce the work done on Hardy-Sobolev spaces and Fock-Sobolev spaces and their operators and operator algebras, including the study of boundedness, compactness, Fredholm property, index theory, spectrum and essential spectrum, norm and essential norm, Schatten-p classes, and the C^* algebras generated by them.

1.1 Hardy-Sobolev space

Let $\mathbb{B}_n$ be the open unit ball in $\mathbb{C}^n$ and $H(\mathbb{B}_n)$ be the space of all holomorphic functions on $\mathbb{B}_n$. For $f \in H(\mathbb{B}_n)$ we use

$$Rf(z)=z_1\frac{\partial f}{\partial z_1}(z)+\cdots+z_n\frac{\partial f}{\partial z_n}(z)$$

to denote the radial derivative of f at z. If

$$f(z)=\sum_{k=0}^{\infty} f_k(z)$$

is the homogeneous expansion of f, then it is easy to see that

$$Rf(z)=\sum_{k=1}^{\infty} k f_k(z).$$

More generally, for any real β and any $f \in H(\mathbb{B}_n)$ with the homogeneous expansion above, we define

$$R^{\beta} f(z)=\sum_{k=1}^{\infty} k^{\beta} f_k(z)$$

and call it the radial derivative of f of order β.