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An Efficient Cartesian Grid-Based Method for Scattering Problems with Inhomogeneous Media

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Abstract. Boundary integral equations provide a powerful tool for the solution of scattering problems. However, often a singular kernel arises, in which case the standard quadratures will give rise to unavoidable deteriorations in numerical precision, thus special treatment is needed to handle the singular behavior. Especially, for inhomogeneous media, it is difficult if not impossible to find out an analytical expression for Green's function. In this paper, an efficient fourth-order accurate Cartesian grid-based method is proposed for the two-dimensional Helmholtz scattering and transmission problems with inhomogeneous media. This method provides an alternative approach to indirect integral evaluation by solving equivalent interface problems on Cartesian grid with a modified fourth-order accurate compact finite difference scheme and a fast Fourier transform preconditioned conjugate gradient (FFT-PCG) solver. A remarkable point of this method is that there is no need to know analytical expressions for Green's function. Numerical experiments are provided to demonstrate the advantage of the current approach, including its simplicity in implementation, its high accuracy and efficiency.

AMS subject classifications: 35J25, 65N30, 65N50, 65N55
Key words: Transmission problem, inhomogeneous media, Cartesian grid-based method, modified fourth-order compact difference scheme, fast Fourier transform preconditioned conjugate gradient solver.

1. Introduction

Wave propagation and scattering phenomena arise from many scientific and engineering applications, such as transport processes in porous or disordered media [18,

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43], biomedicine [49], fluid-solid interaction [20, 60]. The simplest model of a wave scattered by an object is the famous Helmholtz equation. It suffers from considerable mathematical and computational challenges due to the following:

i) The propagation domain is generally very large (even irregular) or unbounded.

ii) The corresponding discrete system is not positive definite.

iii) The solution is highly oscillatory when the wave number is large.

This paper is focused on an efficient Cartesian grid-based method to overcome the first difficulty for the classical two-dimensional exterior Helmholtz problem and acoustic transmission problem.

Over the past few decades, there has been extensive research devoted to dealing with wave scattering in unbounded domains [9, 27, 34, 54, 61, 69]. One of the most popularly used methods is to introduce artificial boundaries enclosing the obstacle, then classical numerical methods can be employed for discretization on the bounded domain between the artificial boundary and the obstacle. The key step of this technique is to construct an artificial boundary condition, which has been addressed by some commonly used classical approaches, such as Dirichlet-to-Neumann (DtN) mapping [15, 19, 28, 32], perfectly matched layer (PML) method [5, 6, 10, 11], Bayliss-Gunzburger-Turkel (BGT) approach [1, 41, 42, 67]. Another conventional methodology is to reformulate problems into boundary integral equations (BIEs) based on Green's formulation [14,24,30,35,44]. In this formulation, the scattering problem defined over an unbounded domain can be transformed into a problem defined on the boundary of the scattering obstacle, thus the dimensionality of the model is reduced by one. This is very beneficial to problems with external (irregular) domain, since it avoids generation of unstructured grids and requires less computer memory.

In spite of the reduced dimensionality of the model and the ease of complex boundary capturing, the classical boundary integral method (BIM) also has some obvious drawbacks, such as the density of resulted linear systems, the dependency on analytical expressions of Green's function, the impossibility of dealing with nonlinear problems or variable coefficient problems and so on. Actually, Green's function involved in the BIEs has a $1/r$ divergence and the integral over the $1/r$ divergence gives rise to singularity. In order to deal with the singular behavior, much effort has been made to achieve high order accuracy. For example, using local change of variable and computing the integral by special Gaussian quadrature directly, separating the singular part and computing the singular part analytically, modifying the original boundary integral equation and shifting the derivative from singular kernel function to the density function, one can see [31] for a review. In particular, Beale and his collaborators [2, 4] presented a simple and accurate method to compute the singular integrals with a regularized kernel by adding some corrections, which are found from analysis near singular points. Chan and his collaborators [33, 50, 51] developed some modern ways to remove singularities analytically by subtracting a simper solution of a related problem with an appropriate choice of free parameters in the solution. Elimination of the singularities means the