Commun. Math. Res.
doi: 10.4208/cmr.2022-0034

Vol. **39**, No. 3, pp. 331-341
August 2023

Hilbert-Schmidtness of Submodules in $H^2(\mathbb{D}^2)$ Containing $\theta(z)-\varphi(w)$

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Received 2 July 2022; Accepted 10 September 2022

Abstract. A closed subspace M of the Hardy space $H^2(\mathbb{D}^2)$ over the bidisk is called submodule if it is invariant under multiplication by coordinate functions z and w. Whether every finitely generated submodule is Hilbert-Schmidt is an unsolved problem. This paper proves that every finitely generated submodule M containing $\theta(z)-\varphi(w)$ is Hilbert-Schmidt, where $\theta(z),\varphi(w)$ are two finite Blaschke products.

AMS subject classifications: Primary 43A65, 47A13

Key words: Hardy space over the bidisk, Hilbert-Schmidt submodule, fringe operator, Fredholm index.

1 Introduction

The Hardy space $H^2(\mathbb{D}^2)$ over the bidisk is a module over the polynomial ring $\mathbb{C}[z,w]$ with module actions defined by multiplication of functions. Since the 60's, a great amount of work has been done to elucidate the structure of the submodules in $H^2(\mathbb{D}^2)$. The interests on the submodules are motivated mainly by three factors. The first, submodules in the one variable Hardy space $H^2(\mathbb{D})$ have a very clean and useful description by the Beurling's theorem; the second, submodules in $H^2(\mathbb{D}^2)$ have a close connection with the functional model theory for single operators [11]; and the third, submodules in $H^2(\mathbb{D}^2)$ provide a huge

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amount of interesting two variable examples, from which some new ideas and techniques for multivariable operator theory can be tested. However, as manifested by Rudin [6], the structure of submodules in $H^2(\mathbb{D}^2)$ is far more complex, and the tools employed in the study of $H^2(\mathbb{D})$ can almost never be effectively carried over to the two variable setting.

The research on $H^2(\mathbb{D}^2)$ is ongoing. One approach to this problem is to study some relatively simple submodules, and hope that the study will generate concepts and techniques for the general picture. In [1], the authors developed a machinery to study multiplication operators on the Bergman space via the quotient module $[z-w]^\perp$. Moreover, by using this machinery, Sun and Zheng [8] gave a new proof of the Beurling-type theorem for the Bergman space. This machinery has been shown to be very useful in dealing with some question on Bergman space so far. Izuchi and Yang [3,4] presented an analysis of the N_φ-type quotient module $[z-\varphi(w)]^\perp$, and they look closely at the case when φ is an inner function. Recently, Zou *et al.* [15] study the submodule $M_{\theta,\varphi}=[\theta(z)-\varphi(w)]$ and the corresponding quotient module $N_{\theta,\varphi}$, where θ,φ are inner functions. They calculate the Hilbert-Schmidt norm of the core operator and the commutators $[S_z^*,S_z]$, $[S_w^*,S_w]$ and $[S_z^*,S_w]$.

Suppose M is a submodule of $H^2(\mathbb{D}^2)$, we denote by R_z,R_w and S_z,S_w the restriction of the multiplication operators M_z,M_w on M and respectively the compression of M_z,M_w on $M^\perp=H^2(\mathbb{D}^2)\ominus M$. In analogy with the operators R_z and S_z on $H^2(\mathbb{D})$, we are interested in the operator pairs (R_z,R_w) and (S_z,S_w) on $H^2(\mathbb{D}^2)$. It is clear that (R_z,R_w) is a pair of commuting isometries, and (S_z,S_w) is a pair of commuting contractions. These pairs encode much information about M and they are the subjects of many recent studies. Let

$$C_M=I-R_zR_z^*-R_wR_w^*+R_zR_wR_z^*R_w^*.$$

C_M is called the core operator or defect operator for M (cf. [2]). M is called a Hilbert-Schmidt submodule if the core operator C_M is Hilbert-Schmidt. Hilbert-Schmidt submodules have many good properties and have been studied extensively in the literature, see [9–14] and the references there in. In particular, it was shown in [13] that C^2 is unitarily equivalent to

$$\begin{pmatrix} [R_z^*,R_z][R_w^*,R_w][R_z^*,R_z] & 0 \\ 0 & [R_z^*,R_w][R_w^*,R_z] \end{pmatrix}.$$

This implies that C_M is Hilbert-Schmidt if and only if $[R_z^*,R_z][R_w^*,R_w]$ and $[R_z^*,R_w]$ are both Hilbert-Schmidt. It is known that if C_M is Hilbert-Schmidt, then the pairs (R_z,R_w) and (S_z,S_w) are Fredholm. Almost all known examples of submodules are Hilbert-Schmidt. The only known non-Hilbert-Schmidt submodule