Commun. Math. Res.
doi: 10.4208/cmr.2022-0047

Vol. **39**, No. 3, pp. 437-475
August 2023

Uniform Convergence of Multigrid V-Cycle on Adaptively Refined Finite Element Meshes for Elliptic Problems with Discontinuous Coefficients

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Received 11 August 2022; Accepted 8 January 2023

Abstract. The multigrid V-cycle methods for adaptive finite element discretizations of two-dimensional elliptic problems with discontinuous coefficients are considered. Under the conditions that the coefficient is quasi-monotone up to a constant and the meshes are locally refined by using the newest vertex bisection algorithm, some uniform convergence results are proved for the standard multigrid V-cycle algorithm with Gauss-Seidel relaxations performed only on new nodes and their immediate neighbours. The multigrid V-cycle algorithm uses $\mathcal{O}(N)$ operations per iteration and is optimal.

AMS subject classifications: 65N55, 65N12
Key words: Multigrid, adaptive finite elements, elliptic problems, discontinuous coefficients, uniform convergence.

1 Introduction

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with possibly reentrant corners. Consider the variational problem of finding $u\in H_0^1(\Omega)$ such that

$$A(u,v)=F(v),\quad \forall v\in H_0^1(\Omega), \tag{1.1}$$

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where $F\in H^{-1}(\Omega)$ and

$$A(u,v)=\int_\Omega a(x)\nabla u\cdot\nabla v\mathrm{d}x,\quad \forall u,v\in H_0^1(\Omega).$$

Here $a(x)$ is a positive piecewise constant function.

It is well known that the reentrant corners and the discontinuous coefficient may cause local singularities of the solution of the elliptic problem. So it is natural to apply the adaptive finite element methods to capture the singularities efficiently. Ever since the pioneering work of [3], the adaptive finite element methods based on a posteriori error estimates have become a central theme in scientific and engineering computations. Recent studies (cf., e.g., [13, 16, 21, 22, 29, 32]) indicate that for appropriately designed adaptive finite element algorithms, the meshes and the associated numerical complexity are quasi-optimal in the sense that, for any elliptic problem, with or without singularities, an appropriately designed adaptive finite element algorithm converges at the same rate with respect to the number of degrees of freedom.

Let $\{\mathcal{M}_j\}_{j=0}^J$ be a family of nested finite element meshes of Ω, let $\mathcal{N}_j$ be the set of interior nodes of $\mathcal{M}_j$, and let $X_j\subset H_0^1(\Omega)$ be the piecewise linear finite element space on $\mathcal{M}_j$ with dimension n_j. The multigrid methods have been proved to be optimal for solving the linear systems from finite element discretizations of the elliptic problems on quasi-uniform meshes (cf., e.g., [6,33], and the references therein). Here "optimal" means that one step of multigrid iteration can reduce the norm of the approximation error by a factor which is bounded away from 1 and independent of n_J, the size of the linear system at level J, while using only $\mathcal{O}(n_J)$ operations. One distinct feature of the locally refined meshes $\mathcal{M}_j, 0\le j\le J$ is that the number of nodes n_J may not grow exponentially with respect to the number of mesh refinements J. As a consequence, when applying traditional multigrid method which performs relaxations on all nodes of $\mathcal{M}_j$, the number of computations for each multigrid iteration can be as bad as $\mathcal{O}(n_J^2)$.

To reduce the computational cost, various local relaxation schemes are proposed for multigrid methods on locally refined meshes (see, e.g., [1, 2, 4, 7, 9, 11, 14, 15, 19, 20, 31], and the references therein). In particular, [31] considered the second-order elliptic problems with smooth coefficients and gave a rigorous convergence analysis for the multigrid V-cycle method. Their theory is uniform with respect to J and applies to any practical adaptive finite element algorithms which use the newest vertex bisection (NVB) strategy for mesh refinements. It is shown that performing local relaxations at new nodes and their immediately neighboring nodes can already guarantee the uniform convergence of the multigrid algorithm. Here a immediately neighboring node of one new node means that they