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Inverse Scattering Method for Constructing Multisoliton Solutions of Higher-Order Nonlinear Schrödinger Equations

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Abstract. We develop an inverse scattering method for an integrable higher-order nonlinear Schrödinger equation (NLSE) with the zero boundary condition at the infinity. An appropriate Riemann-Hilbert problem is related to two cases of scattering data — viz. for N simple poles and a one higher-order pole. This allows to obtain the exact formulae of N-th order position and soliton solutions in the form of determinants. In addition, special choices of free parameters allow to determine remarkable characteristics of these solutions and to discuss them graphically. The results can be also applied to other types of NLSEs such as the standard NLSE, Hirota equation, and complex modified KdV equation. They can help to further explore and enrich related nonlinear wave phenomena.

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Key words: Inverse scattering method, Riemann-Hilbert problem, soliton.

1. Introduction

It is well-known that the study of integrable systems plays a vital role in mathematical physics. In particular, the standard nonlinear Schrödinger equation (NLSE)

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0 \tag{1.1}$$

is an important integrable system because of its rich mathematical structure and physical significance. There are many physical contexts where the NLSE appears. For example, the NLSE (1.1) describes the weakly nonlinear surface wave in deep water. More importantly, the NLSE (1.1) models the soliton propagation in optical fibers where only the group velocity dispersion and the self-phase modulation effects are considered. However, for ultrashort pulse in optical fibers, the higher-order dispersion, self-steepening, higher-order

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nonlinearity and self-frequency shift should be taken into account. Thus when studying high-speed optical fiber transmission systems, the NLSE has to be complemented by appropriate higher-order terms. In recent years, the NLSEs with higher-order terms have been actively investigated [6, 9, 11, 13, 35, 40]. However, in this work we do not study a special NLSE. Instead, we go beyond the standard NLSE (1.1) by introducing the generalized infinite NLSE hierarchy [8]

$$i\psi_t+\sum_{j=1}^{\infty}\left(\alpha_{2j}K_{2j}-i\alpha_{2j+1}K_{2j+1}\right)=0, \tag{1.2}$$

where each coefficient $\alpha_j, j=1,2,\dots$ is a free real number, $\psi(t,x)$ is the complex field, t and x represent longitudinal and transverse variables, respectively. The Eq. (1.2) is a crucial integrable extension of NLSEs up to infinite order [7, 8, 24, 27]. In particular, we have:

1) $K_2(\psi)$ is the second-order NLSE term [46], viz.

$$K_2(\psi)=\psi_{xx}+2|\psi|^2\psi.$$

2) $K_3(\psi)$ is the third-order term with third-order dispersion [20], viz.

$$K_3(\psi)=\psi_{xxx}+6|\psi|^2\psi_x.$$

3) $K_4(\psi)$ is the fourth-order term with fourth-order dispersion [33], viz.

$$K_4(\psi)=\psi_{xxxx}+6\psi_x^2\psi^*+4|\psi_x|^2\psi+8|\psi|^2\psi_{xx}+2\psi^2\psi_{xx}^*+6|\psi|^4\psi.$$

4) $K_5(\psi)$ is the quintic term with fifth-order dispersion [22], viz.

$$K_5(\psi)=\psi_{xxxxx}+10|\psi|^2\psi_{xxx}+10\left(\psi|\psi_x|^2\right)_x+20\psi^*\psi_x\psi_{xx}+30|\psi|^4\psi_x. \tag{1.3}$$

The following higher-order terms are presented in [8]. To determine common and different features of even and odd terms, individually or in combination with the fundamental NLSE term, we first consider the first five terms (also called higher-order NLSE) and then it can be extended the conclusion into the infinite NLSE (1.2). In the past few years, many efforts were devoted to studying exact solutions of nonlinear integrable equations. For example, Ma [31] has proposed a generalized Wronskian method to construct exact solutions of certain nonlinear integrable equations. In [16], the authors have investigated the rogue waves for a mixed coupled nonlinear Schrödinger equation via Darboux-Dressing transformation. In [44], we have provided the characteristics of the breather and rogue waves in a (2+1)-dimensional nonlinear integrable equation. However, there are only a few studies devoted to exact solutions of the nonlinear integrable equations of higher-orders.

The solutions of nonlinear integrable equations are interesting objects from both mathematical and physical point of view. Therefore, over the years various methods have been proposed for their solution, including the Darboux transformation (DT) [32], inverse scattering transform (IST) [1], unified transform [17], nonlinear steepest descent method [15],