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High-Order Multi-Resolution Weno Schemes with Lax-Wendroff Method for Fractional Differential Equations

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Abstract. New high-order finite difference multi-resolution weighted essentially non-oscillatory (WENO) schemes with the Lax-Wendroff (LW) time discretization method for solving fractional differential equations with the Caputo fractional derivative are considered. The fractional derivative of order $\alpha \in (1,2)$ is split into an integral part and a second derivative term. The Gauss-Jacobi quadrature method is employed to solve the integral part, and a new multi-resolution WENO method for discretizing the second derivative term is developed. High-order spatial reconstruction procedures use any positive numbers (with a minor restriction) as linear weights. The new multi-resolution WENO-LW methods are one-step explicit high-order finite difference schemes. They are more compact than multi-resolution WENO-RK schemes of the same order. The LW time discretization is more cost efficient than the Runge-Kutta time discretization method. The construction of new multi-resolution WENO-LW schemes is simple and easy to generalize to arbitrary high-order accuracy in multi-dimensions. One- and two-dimensional examples with strong discontinuities verify the good performance of new fourth-, sixth-, and eighth-order multi-resolution WENO-LW schemes.

AMS subject classifications: 65M06, 35R11

Key words: Multi-resolution WENO-LW scheme, fractional differential equation, Lax-Wendroff time discretization, Gauss-Jacobi quadrature, high-order accuracy.

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1. Introduction

In order to solve fractional differential equations, we employ high-order finite difference multi-resolution weighted essentially non-oscillatory (WENO) schemes with the Lax-Wendroff time discretization method on structured meshes [21]. This Lax-Wendroff time discretization method is different from classical total variation diminishing (TVD) Runge-Kutta time discretization methods [10, 32]. The Lax-Wendroff time discretization method can save computational cost better than the Runge-Kutta time discretization method. Unlike to many classic WENO schemes [19, 25], we use the information defined on the multi-resolution style hierarchy of the nested spatial stencils. The linear weights in the reconstruction process can be any positive numbers whose sum is one. It is simple and can be easily generalized to arbitrary high-order accuracy schemes in applications. The spatial reconstruction of the new high-order multi-resolution WENO-LW schemes uses only a series of unequal-sized spatial stencils. Besides, the number of spatial stencils is smaller than for the classical finite difference WENO schemes of the same order [11, 19, 31] with the application of the TVD Runge-Kutta time discretization methods [10, 32].

In the development of numerical methods for the convection-diffusion partial differential equations, the high-order WENO scheme has been extensively studied. Liu *et al.* [23] first creatively designed a third-order finite volume WENO scheme on structured meshes in 1994. Two years later, Jiang and Shu [19] designed a multi-dimensional fifth-order finite difference WENO scheme based on the ENO scheme and proposed a construction method of smoothness indicators and nonlinear weights. In 2016, Zhu and Qiu [39] designed a new fifth-order finite difference WENO scheme based on the classic WENO schemes to solve the multi-dimensional hyperbolic conservation laws. At the same time, an additional processing of negative linear weights was avoided, which greatly simplified the calculation procedure. Two years later, Zhu and Shu [41] improved this method and proposed a new type of multi-resolution WENO scheme that can achieve arbitrary high-order accuracy in smooth regions. Up to now, more and more classic high-order numerical schemes have been proposed in different fields [5, 13, 24, 26, 30, 33, 34, 36, 37].

Fractional differential equations are important research topics in physics, environment, hydrology, biology, and other disciplines. In particular, it is exploited in the diffusion of water or pollutants in soil, turbulent eddy motion, and neural signal transmission. In recent years, the construction of numerical schemes for fractional differential equations receives growing attention. In 2004, Liu *et al.* [22] introduced the lines method to discretize the space fractional differential equation. Meerschaert and Tadjeran [27] solved one-dimensional fractional equations by finite difference schemes. In 2013, Chen and Deng [9] proposed an alternating directions implicit method for the space fractional equation. In 2020, Bouharguane and Seloula [7] proposed a local discontinuous Galerkin method for solving the convection-diffusion-fractional anti-diffusion equations. However, most of these numerical studies are not suited for continuous fractional differential equations and there are only a few studies for fractional differential equations with discontinuous initial values. In 2013, Deng *et al.* [11] proposed a high-order finite difference WENO scheme to discretize fractional differential equations whose initial values may be discontinuous. It is