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# A New Family of Nonconforming Elements with $\boldsymbol{H}(\mathrm{curl})$-Continuity for the 3D Quad-Curl Problem

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**Abstract.** We propose and analyze a new family of nonconforming finite elements for the three-dimensional quad-curl problem. The proposed finite element spaces are subspaces of $\boldsymbol{H}(\mathrm{curl})$, but not of $\boldsymbol{H}(\mathrm{grad\ curl})$, which are different from the existing nonconforming ones [10,12,13]. The well-posedness of the discrete problem is proved and optimal error estimates in discrete $\boldsymbol{H}(\mathrm{grad\ curl})$ norm, $\boldsymbol{H}(\mathrm{curl})$ norm and $\boldsymbol{L}^2$ norm are derived. Numerical experiments are provided to illustrate the good performance of the method and confirm our theoretical predictions.

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**Key words**: Quad-curl problem, nonconforming finite element method.

## 1 Introduction

The quad-curl problem arises in many areas such as inverse electromagnetic scattering [4, 15, 16] and magnetohydrodynamics [23]. Finite element methods (FEMs) are natural choices for numerical treatment of such problems. In recent years, various FEMs have been proposed and analyzed.

Among various FEMs, conforming elements are natural candidate, for their error estimates can be obtained by the standard framework. Conforming methods on triangles and rectangles in two dimensions were devised in [11,20]. Then a family of curl-curl conforming elements on tetrahedra was constructed in [21], which has at least 315 degrees of freedom (DOFs) on each tetrahedron. Later, by enriching the shape function space with macro-element bubble functions, the authors of [12] constructed a family of conforming elements whose lowest order element has only 18 DOFs. However, the use of

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macro-element bubble functions adds some difficulty in coding compared with the finite element space using only pure polynomials.

To avoid the use of conforming finite elements, nonconforming FEMs were studied in [13,23]. These methods have low computational cost since they have small number of degrees of freedom (DOFs), but both of them are low-order. The authors of [12] proposed another $\boldsymbol{H}(\text{grad curl})$-nonconforming elements, but macro-element bubble functions are still required.

Another approach to avoid the use of conforming finite elements is to write the quad-curl problem as a system of second-order problems and use the mixed FEM [17]. Different mixed schemes for the quad-curl problems were presented in [22]. Instead of solving the quad-curl problem directly, finite element methods based on decoupling of mixed formulation were studied in [3,5].

In addition to the above methods, discontinuous Galerkin (DG) methods are also used to deal with the operator $(\nabla\times)^4$. In [10], a DG method based on $H(\text{curl})$-conforming element was presented for the quad-curl problem. A hybridizable discontinuous Galerkin (HDG) method was investigated in [6]. Recently, a $C^0$ interior penalty method was studied in [18]. However, these methods modify the variational formulation to tackle discontinuity of the basis functions, which increases the difficulty of programming to a certain extent.

The main purpose of this work is to construct a family of nonconforming elements which uses only pure polynomials as shape functions, allows high order extension, and is easier to code than conforming, DG and HDG methods. By comparison, we find that the nonconforming space $\boldsymbol{U}_h$ satisfies $\boldsymbol{U}_h \not\subseteq \boldsymbol{H}(\text{curl})$, $\nabla_h\times\boldsymbol{U}_h \not\subseteq \boldsymbol{H}^1$ in [13,23], and satisfies $\boldsymbol{U}_h \not\subseteq \boldsymbol{H}(\text{curl})$, $\nabla_h\times\boldsymbol{U}_h \subset \boldsymbol{H}^1$ in [12]. A natural question is whether there is a space that satisfies $\boldsymbol{U}_h \subset \boldsymbol{H}(\text{curl})$, $\nabla\times\boldsymbol{U}_h \not\subseteq \boldsymbol{H}^1$, since for $\boldsymbol{u}_h$ in such a space, we would have $\nabla\times\boldsymbol{u}_h \in \boldsymbol{H}(\text{div})$, $\nabla\times\boldsymbol{u}_h$ is divergence-free and has some tangential continuity. This reminds us of nonconforming elements for the Stokes and Brinkman problems [9,19]. The main idea to construct nonconforming elements in [9] is to modify $\boldsymbol{H}(\text{div})$-conforming finite elements to have some tangential continuity. More precisely, their local space on each simplex $K$ is of the form

$$\boldsymbol{M}(K)+\nabla\times(b_K\boldsymbol{Q}(K)),$$

where $\boldsymbol{M}(K)$ is the local $\boldsymbol{H}(\text{div})$-conforming space, $b_K$ is the element bubble function that vanishes on $\partial K$ and the space $\boldsymbol{Q}(K)$ is spanned by the face bubble functions multiplied by some vector-valued polynomials. Inspired by this, we construct our local space in the form

$$\boldsymbol{N}(K)+b_K\boldsymbol{Q}(K),$$

where $\boldsymbol{N}(K)$ is the local $\boldsymbol{H}(\text{curl})$-conforming space. Note that $b_K$ vanishes on $\partial K$ and hence the resulting space is still $\boldsymbol{H}(\text{curl})$-conforming. Thus, the only purpose of adding the function $b_K\boldsymbol{Q}(K)$ is to enforce some tangential continuity of $\nabla\times\boldsymbol{u}_h$. As $\boldsymbol{Q}(K)$ in [9] being arbitrary order, the nonconforming space constructed here can also be arbitrary order.