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Adaptive Ensemble Kalman Inversion with Statistical Linearization

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Abstract. The ensemble Kalman inversion (EKI), inspired by the well-known ensemble Kalman filter, is a derivative-free and parallelizable method for solving inverse problems. The method is appealing for applications in a variety of fields due to its low computational cost and simple implementation. In this paper, we propose an adaptive ensemble Kalman inversion with statistical linearization (AEKI-SL) method for solving inverse problems from a hierarchical Bayesian perspective. Specifically, by adaptively updating the unknown with an EKI and updating the hyper-parameter in the prior model, the method can improve the accuracy of the solutions to the inverse problem. To avoid semi-convergence, we employ Morozov's discrepancy principle as a stopping criterion. Furthermore, we extend the method to simultaneous estimation of noise levels in order to reduce the randomness of artificially ensemble noise levels. The convergence of the hyper-parameter in prior model is investigated theoretically. Numerical experiments show that our proposed methods outperform the traditional EKI and EKI with statistical linearization (EKI-SL) methods.

AMS subject classifications: 62F15, 65C35, 65N21, 65N75

Key words: Ensemble Kalman inversion, statistical linearization, adaptive, Bayesian inverse problem.

1 Introduction

The ensemble Kalman filter (EnKF) [16] is one of the most effective methods for dealing with nonlinear dynamic systems in the field of data assimilation. It is widely applied in a variety fields such as numerical weather prediction [2, 20], reservoir engineering [13] and oceanography [16]. The EnKF method can also be used to solve the general inverse problems, also known as ensemble Kalman inversion (EKI) [1,22,23,31]. It determines the

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next estimate by computing the sample mean and covariance of the empirical ensemble, which can be thought of as a gradient-free black-box optimizer. This method has been successfully applied to various practical applications, including industry and petroleum fields [21], and electrical impedance tomography (EIT) [8].

The ensemble update formula of EKI is equivalent to a coupled system of discrete stochastic differential equations (SDEs), which provides direction for the theoretical analysis of EKI [6,11,15,29,30,33]. Furthermore, the authors of [3–5,14,25] investigated properties under continuous time limits, such as convergence, well-posedness and so on. In contrast, EKI can be viewed as a least-squares method for solving inverse problems from an optimization perspective. Regularization is frequently required because the inverse problem is often ill-posed [9,10,24,34]. EKI has also been demonstrated to be a promising optimization method that has been applied to various machine learning tasks [12,19,27].

Despite the effectiveness of EKI methods in inverse problems, they have 'initial subspace properties' [23] that make the prior model selection critical. The prior models of the most recent algorithms, on the other hand, are pre-determined and do not account for the observation and the efficacy of the obtained solution across the EKI. In this work, we present a novel iterative formulation that combines hierarchical Bayesian theory [26, 32] with statistical linearization techniques [7] to refine the prior model in the ensemble Kalman inversion. Specifically, we update the ensemble while adaptively updating the hyper-parameter in prior based on maximum a posteriori (MAP) estimation. Furthermore, we investigate the convergence of the hyper-parameter theoretically under strict assumptions. To avoid semi-convergence, we include an appropriate stopping criterion in the algorithm. We also propose a generalized adaptive ensemble Kalman inversion with statistical linearization (GAEKI-SL) in combination with hierarchical models for identifying the variance level in the error data while inverting the parameters in the more general case. Finally, we use numerical examples to demonstrate the effectiveness of our framework.

Overall, the main contributions of this paper can be summarized as follows:

- We propose an adaptive ensemble Kalman inversion with statistical linearization (AEKI-SL) method, which significantly improves the accuracy over traditional EKI and EKI with statistical linearization (EKI-SL) methods.

- To determine the noise level of the observation, we generalize the AEKI-SL method to simultaneously adaptively update the noise level, which we call GAEKI-SL. As a result, an approximation closer to the true noise level is obtained, which can be used to define a stopping iteration criterion in EKI framework.

- We theoretically perform a convergence analysis of the AEKI-SL method's hyper-parameter for linear problems.

- We demonstrate the effectiveness of our methods using three examples that include linear and nonlinear PDE inverse problems. It is shown that AEKI-SL can effectively capture the solution structure for all test problems.