Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2022-0311

Vol. **33**, No. 5, pp. 1466-1508
May 2023

# Superconvergence Analysis of $C^m$ Finite Element Methods for Fourth-Order Elliptic Equations I: One Dimensional Case

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Received 19 December 2022; Accepted (in revised version) 26 March 2023

**Abstract.** In this paper, we study three families of $C^m(m=0,1,2)$ finite element methods for one dimensional fourth-order equations. They include $C^0$ and $C^1$ Galerkin methods and a $C^2$-$C^0$ Petrov-Galerkin method. Existence, uniqueness and optimal error estimates of the numerical solution are established. A unified approach is proposed to study the superconvergence property of these methods. We prove that, for $k$th-order elements, the $C^0$ and $C^1$ finite element solutions and their derivative are superconvergent with rate $h^{2k-2}$ $(k\geq 3)$ at all mesh nodes; while the solution of the $C^2$-$C^0$ Petrov-Galerkin method and its first- and second-order derivatives are superconvergent with rate $h^{2k-4}$ $(k\geq 5)$ at all mesh nodes. Furthermore, interior superconvergence points for the $l$-th $(0\leq l\leq m+1)$ derivate approximations are also discovered, which are identified as roots of special Jacobi polynomials, Lobatto points, and Gauss points. As a by-product, we prove that the $C^m$ finite element solution is superconvergent towards a particular Jacobi projection of the exact solution in the $H^l$ $(0\leq l\leq m+1)$ norms. All theoretical findings are confirmed by numerical experiments.

**AMS subject classifications**: 65N30, 65N15

**Key words**: $C^m$ finite element methods, superconvergence, fourth-order elliptic equations.

## 1 Introduction

Fourth-order partial differential equations are important mathematical models and widely used in physics and engineering such as elastic bending problems in structure

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mechanics [29,46] and strain gradient theory [5,45]. Among numerous numerical methods to approximate fourth-order equations and biharmonic problems, the standard $C^1$-conforming finite element method is in common use (see, e.g., [13]). To relax the continuity condition for derivatives of $C^1$-conforming elements, some $C^0$ elements, on the other hand, have been developed. We refer to [10, 41, 49] for the mixed finite element method, and [25] for the interior penalty Galerkin method. Meanwhile, non-conforming finite elements, e.g., the Adini element [1] and the Morley element [40, 44], have also found their applications in solving fourth-order equations, due to their flexibility of handling high order derivatives. In addition, discontinuous Galerkin (DG) methods have been proposed for solving time-dependent fourth-order equations [39, 52], fourth-order boundary value problems [8, 32] and biharmonic equations [27]. More recently, hybrid high-order methods, which can be embedded into the broad framework of hybridizable DG methods, have also been successfully implemented for fourth-order problems. We refer to [11,24] for this line of research.

Despite rich literature on numerical methods for forth-order equations, however, the superconvergence study for those methods is still far from developed. Superconvergence is a hot topic and has attracted much attention in scientific and engineering computations. For the past several decades, the superconvergence theory has been well established for the classical $C^0$-conforming finite element method (see, e.g., [7, 12, 19, 20, 26, 33, 34, 36, 42, 47, 48, 57] ), the $C^0$ finite volume method (see, e.g., [14–16,23,51]), the spectral Galerkin method (see, e.g., [54,55]), and the DG method (see, e.g., [2–4, 21, 22, 28, 50, 53, 56]). Most of these superconvergence results were developed for second-order elliptic problems while the relevant work for fourth-order equations is far from complete. In [35] and [6], the authors studied the superconvergence of the Ciarlet-Raviart method and the Hellan-Herrmann-Johnson method for the biharmonic equation, respectively. Superconvergence properties for the Morley element were also investigated in [30,31]. In [9], the authors established superconvergence results of the ultra weak discontinuous Galerkin method for fourth-order boundary value problems. As for superconvergence of $C^m$ finite element methods for fourth-order equations, the only result we know of is in [19], where the author proved that both the function value and its first-order derivative approximations of the $C^1$-conforming solution are superconvergent at mesh nodes with order $h^{2k-2}$.

This paper is the first one in the series of superconvergence analysis of $C^m$ $(m=0,1,2)$ finite element methods for fourth-order problems, where one dimensional equations are under concerned. The $C^1$ method we adopt in this study is the standard $C^1$-$C^1$ conforming finite element method, here by $C^m$-$C^n$ we mean that the trial space is taken as $C^m$ element while the test space is chosen as $C^n$ element. The $C^0$ method we concerned is the $C^0$-$C^0$ Galerkin method which belongs to the category of the interior penalty Galerkin method, where the penalty term and coefficients need to be selected to ensure the stability or well-posedness of the solution. For the purpose of superconvergence, the penalty term is specially designed in our $C^0$-$C^0$ Galerkin schemes. The proposed $C^2$-$C^0$ Petrov-Galerkin scheme for fourth-order equations is brand new, which provides a better ap-