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Decay of the Compressible Navier-Stokes Equations with Hyperbolic Heat Conduction

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Abstract. The global solution to the Cauchy problem of the compressible Navier-Stokes equations with hyperbolic heat conduction in dimension three is constructed when the initial data in H^3 norm is small. By using several elaborate energy functionals together with the interpolation trick, we simultaneously obtain the optimal L^2-decay estimate of the solution and its derivatives when the initial data is bounded in negative Sobolev (Besov) space or $L^1(\mathbb{R}^3)$. Specially speaking, the fluid density, the fluid velocity and the fluid temperature in L^2-norm have the same decay rate as the Navier-Stokes-Fourier equations, while the flux q has faster L^2-decay rate as $(1+t)^{-2}$. Our proof is based on a family of scaled energy estimates with minimum derivative counts and interpolations among them without linear decay analysis for a 8×8 Green matrix of the system. To the best of our knowledge, it is the first result on the large time behavior of this system.

AMS subject classifications: 35B40, 35A09, 35Q35

Key words: Decay rate, Navier-Stokes equations, hyperbolic heat conduction, energy method.

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1 Introduction

The compressible Navier-Stokes equations with hyperbolic heat conduction [13] takes the following form:

$$\partial_t\rho+\mathrm{div}(\rho u)=0, \tag{1.1a}$$

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$$\partial_t(\rho u)+\mathrm{div}(\rho u\otimes u)+\nabla P=\mathrm{div}S, \tag{1.1b}$$

$$\partial_t\left(\rho\left(e+\frac{1}{2}u^2\right)\right)+\mathrm{div}\left(\rho u\left(e+\frac{1}{2}u^2\right)+uP\right)+\mathrm{div}q=\mathrm{div}(uS), \tag{1.1c}$$

$$\tau\partial_t q+q+\kappa\nabla\theta=0, \tag{1.1d}$$

where the unknown functions $\rho, u=(u_1,u_2,\cdots,u_n), P, S, e, q$ represent fluid density, velocity, pressure, stress tensor, specific internal energy per unit mass and flux, respectively. The Eq. (1.1d) represents Cattaneo law (Maxwell law, etc.), and $\tau>0$ is the constant relaxation time and $\kappa>0$ is the constant heat conductivity. Here we assume the fluid to be a Newtonian fluid, that is, $S=\mu(\nabla u+(\nabla u)^T)+\mu'\mathrm{div}uI$, where μ and μ' are the coefficient of viscosity and the second coefficient of viscosity, respectively, satisfying $\mu>0$, $\mu'+2\mu/n\geq 0$.

In this paper, we will study the global existence and large time behavior of the smooth solutions for the system (1.1) with the following initial data:

$$\rho(x,0)=\rho_0(x)>0, \quad u(x,0)=u_0(x), \quad \theta(x,0)=\theta_0(x)>0, \quad q(x,0)=q_0(x). \tag{1.2}$$

Here we consider the general equations of state and assume that the pressure $P(\rho,\theta)$ and $e=e(\rho,\theta)$ are smooth function of (ρ,θ) satisfying

$$\rho^2 e_\rho(\rho,\theta)=P(\rho,\theta)-\theta P_\theta(\rho,\theta), \tag{1.3}$$

where θ is the absolute temperature. Obviously, our assumption includes the case of a polytropic gas $P=R\rho\theta$, $e=c_v\theta$.

When $\tau=0$, the system (1.1) is the classical full compressible Navier-Stokes equations, in which the relation between the heat flux and the temperature represents Fourier law, $q=-\kappa\nabla\theta$. Due to its importance for both physical and mathematical applications, the well-posed theory has been widely studied for the system combined with Fourier law, or the isentropic case, see [1,7,8,10,14–18] and references therein.

In the following, we mainly review some results on the decay rate of the closely related models. A lot of works have been done on the existence, stability and L^p-decay rates with $p\geq 2$ for either isentropic or non-isentropic (heat-conductive) cases, cf. [5,6,21–23] in various settings by using (weighted) energy method together with spectrum analysis. Recently, Danchin and Xu [2] developed optimal decay rate in general critical spaces and any dimension $n\geq 2$ under a mild additional decay assumption that is satisfied if the low frequencies of the initial data. On the other hand, Liu and Zeng [20] first studied the pointwise estimates of solution to general hyperbolic-parabolic systems in dimension one by using the method of Green function. Hoff and Zumbrun [11] investi-