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The Euler Limit of the Relativistic Boltzmann Equation

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Abstract. In this work we prove the existence and uniqueness theorems of the solutions to the relativistic Boltzmann equation for analytic initial fluctuations on a time interval independent of the Knudsen number $\epsilon>0$. As $\epsilon\to 0$, we prove that the solution of the relativistic Boltzmann equation tends to the local relativistic Maxwellian, whose fluid-dynamical parameters solve the relativistic Euler equations and the convergence rate is also obtained. Due to this convergence rate, the Hilbert expansion is verified in the short time interval for the relativistic Boltzmann equation. We also consider the physically important initial layer problem. As a by-product, an existence theorem for the relativistic Euler equations without the assumption of the non-vacuum fluid states is obtained.

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1 Introduction

The relativistic Boltzmann equation, which is a fundamental model describing the motion of fast moving particles in kinetic theory, takes the form of

$$P\otimes\partial_X F=-\frac{1}{\epsilon}\mathcal{C}(F,F). \tag{1.1}$$

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Here $\otimes$ represents the Lorentz inner product $(+---)$ of 4-vectors. As is customary we write $X=(x_0,x)$ with $x\in\mathbb{R}^3$ and $x_0=-t$, and $P=(p_0,p)$ with momentum $p\in\mathbb{R}^3$ and energy $p_0=\sqrt{c^2+|p|^2}$, where c denotes the speed of light. For convenience of presentation, we rewrite (1.1) as

$$\partial_t F+\hat{p}\cdot\nabla_x F=\frac{1}{\epsilon}\mathcal{Q}(F,F) \tag{1.2}$$

with $\mathcal{Q}(F,F)=\mathcal{C}(F,F)/p_0$, where the unknown $F=F^{\epsilon}(t,x,p)$ stands for the density distribution function of time $t\geq 0$, space $x\in\mathbb{R}^3$ and momentum $p\in\mathbb{R}^3$ and the dimensionless parameter ϵ is the Knudsen number, which is the ratio of the particle mean free path to a characteristic physical length scale. Here the dot represents the standard Euclidean dot product, and the normalized velocity of a particle is denoted as

$$\hat{p}=c\frac{p}{p_0}=\frac{p}{\sqrt{1+|p|^2/c^2}}.$$

For notational simplicity we normalize all the physical constants to be one. Then

$$p_0=\sqrt{1+|p|^2},\quad \hat{p}=\frac{p}{p_0}. \tag{1.3}$$

We rewrite (1.2) supplemented with initial data as

$$\partial_t F+\hat{p}\cdot\nabla_x F=\frac{1}{\epsilon}\mathcal{Q}(F,F),\quad F(0,x,p)=F_0(x,p). \tag{1.4}$$

To describe the relativistic Boltzmann collision term, we introduce the relative momentum g as

$$g=g(p,q)=\sqrt{2(p_0q_0-p\cdot q-1)} \tag{1.5}$$

and also the quantity s as

$$s=s(p,q)=g^2+4=2(p_0q_0-p\cdot q+1). \tag{1.6}$$

Note that $s=g^2+4$ and this may differer from that in [19] by a constant factor. The Mϕller velocity is given by

$$v_\phi=v_\phi(p,q)=\sqrt{\left|\frac{p}{p_0}-\frac{q}{q_0}\right|^2-\left|\frac{p}{p_0}\times\frac{q}{q_0}\right|^2}=\frac{g\sqrt{s}}{2p_0q_0}. \tag{1.7}$$

Then we may express the collision operator $\mathcal{Q}(F,G)$ in the form (see [11,14,19])

$$\mathcal{Q}(F,G)=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} v_\phi\sigma(g,\theta)\left[F(p')G(q')-F(p)G(q)\right]dq\,d\omega, \tag{1.8}$$