JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **36**, No. 2, pp. 147-190

doi: 10.4208/jpde.v36.n2.3
June 2023

A New Regularization Method for a Parameter Identification Problem in a Non-linear Partial Differential Equation

NAIR M. Thamban[1,*] and ROY Samprita Das[2]

[1] *Department of Mathematics, BITS Pilani, K K Birla Goa Campus, Zuarinager, Goa 403726, India.*
[2] *Department of Mathematics and Statistics, IISER Kokota, Nadia, West Bengal 741246, India.*

Received 25 May 2021; Accepted 19 April 2022

Abstract. We consider a parameter identification problem associated with a quasi-linear elliptic Neumann boundary value problem involving a parameter function $a(\cdot)$ and the solution $u(\cdot)$, where the problem is to identify $a(\cdot)$ on an interval $I:=g(\Gamma)$ from the knowledge of the solution $u(\cdot)$ as g on Γ, where Γ is a given curve on the boundary of the domain $\Omega\subseteq\mathbb{R}^3$ of the problem and g is a continuous function. The inverse problem is formulated as a problem of solving an operator equation involving a compact operator depending on the data, and for obtaining stable approximate solutions under noisy data, a new regularization method is considered. The derived error estimates are similar to, and in certain cases better than, the classical Tikhonov regularization considered in the literature in recent past.

AMS Subject Classifications: 35R30, 65N30, 65J15, 65J20, 76S05

Chinese Library Classifications: O175.29

Key Words: Ill-posed; regularization; parameter identification.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with $C^{1,1}$ boundary. Consider the problem of finding a weak solution $u\in H^1(\Omega)$ of the partial differential equation

$$-\nabla\cdot(a(u)\nabla u)=0,\quad \text{in } \Omega \tag{1.1}$$

*Corresponding author. *Email addresses:* `mtnair@faculty.iitm.ac.in`, `mtnair@goa.bits-pilani.ac.in` (M. T. Nair), `samprita.dasroy@gmail.com` (S. D. Roy)

with boundary condition

$$a(u)\frac{\partial u}{\partial \nu}=j, \quad \text{on } \partial\Omega, \tag{1.2}$$

where $a\in H^1(\mathbb{R})$ and $j\in L^2(\partial\Omega)$. One can come across this type of problems in the steady state heat transfer problem with u being the temperature, a the thermal conductivity which is a function of the temperature, and j the heat flux applied to the surface. In this regard, the following result is known (see [1–3]):

Theorem 1.1. *Let $a\geq\kappa_0>0$ a.e. for some constant κ_0 and $\int_{\partial\Omega} j=0$. Then there exists $u\in H^1(\Omega)$ such that* (1.1) *and* (1.2) *are satisfied. If, in addition, $j\in W^{(1-1/p),p}(\partial\Omega)$ with $p>3$, then $u\in C^1(\overline{\Omega})$.*

In view of the above theorem, we assume that,

$$j\in W^{(1-1/p),p}(\partial\Omega) \quad \text{such that} \quad \int_{\partial\Omega} j=0 \quad \text{for some } p>3. \tag{1.3}$$

Suppose $\gamma:[0,1]\to\partial\Omega$ is a C^1- curve on $\partial\Omega$ and $g:\Gamma\to\mathbb{R}$ such that $g\circ\gamma\in C^1([0,1])$, where Γ is the range of γ. One of the inverse problems associated with (1.1)-(1.2) is:

> **Problem (P):** *To identify an $a\in H^1(\mathbb{R})$ on $I:=g(\Gamma)$ such that the corresponding u satisfies* (1.1)-(1.2) *along with the requirement*
>
> $$u=g \quad \text{on } \Gamma. \tag{1.4}$$

In the following we shall use the same notation for $a\in H^1(\mathbb{R})$ and for its restriction on I as a function in $H^1(I)$.

We shall see that the Problem (P) is ill-posed, in the sense that the solution $a_{|_I}$ does not depend continuously on the data g and j (see Section 2). To obtain a stable approximate solution for the Problem (P), we use a new regularization method which is different from some of the standard ones in the literature. We discuss this method in Section 3.

The existence and uniqueness of solution for the Problem (P) is known under some additional conditions on γ and g, as specified in Section 2 (see, e.g., [3, 4]). In [2] and [3] the problem of finding a stable approximate solution of the problem is studied by employing Tikhonov regularization with noisy data. In [2], with the noisy data g^δ, in place of g, satisfying $\|g-g^\delta\|_{L^2(\Gamma)}\leq\delta$, convergence rate $\|a-a^\delta\|_{H^1(I)}=O(\sqrt{\delta})$ is obtained whenever $a\in H^4(I)$ and its trace is Lipschitz on $\partial\Omega$, where a^δ is the approximate solution obtained via Tikhonov regularization. In [3], the rate $\|a-a^\delta\|_{L^2(I)}=O(\sqrt{\delta})$ is obtained without the additional assumption on a, where noise in j as well as g is also considered as

$$\|j-j^\delta\|_{L^2(\partial\Omega)}\leq\delta, \qquad \|g-g^\delta\|_{W^{1,\infty}(\Gamma)}\leq\delta. \tag{1.5}$$

It is stated in [3] that "the rate $O(\sqrt{\delta})$ is possible with respect to H^1-norm, provided some additional smoothness conditions are satisfied"; however, the details of the analysis is missing.