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The A Priori and A Posteriori Error Estimates for Modified Interior Transmission Eigenvalue Problem in Inverse Scattering

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Abstract. In this paper, we discuss the conforming finite element method for a modified interior transmission eigenvalues problem. We present a complete theoretical analysis for the method, including the a priori and a posteriori error estimates. The theoretical analysis is conducted under the assumption of low regularity on the solution. We prove the reliability and efficiency of the a posteriori error estimators for eigenfunctions up to higher order terms, and we also analyze the reliability of estimators for eigenvalues. Finally, we report numerical experiments to show that our posteriori error estimator is effective and the approximations can reach the optimal convergence order. The numerical results also indicate that the conforming finite element eigenvalues approximate the exact ones from below, and there exists a monotonic relationship between the conforming finite element eigenvalues and the refractive index through numerical experiments.

AMS subject classifications: 65N25, 65N30

Key words: Modified interior transmission eigenvalues, a priori error estimates, a posteriori error estimates, adaptive algorithm.

1 Introduction

The eigenvalue problems in inverse scattering and its numerical methods are one of the topics of academic concern (see [1–4]). Among them, due to the important applications in estimating the properties of scattering materials and the theoretical research value of

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uniqueness and reconstruction of inverse scattering theory, the research on numerical methods for transmission eigenvalues has achieved relatively rich results, for example, see [5–14] and so on. However, the method of using transmission eigenvalues as the target characteristic is only applied to non-absorbing media or materials with low absorption rates. So, after modifying the far-field operator in inverse scattering theory, the Steklov eigenvalue problem and the modified interior transmission eigenvalue problem (mITEP) which can be applied to absorbing media are introduced (cf. [15–17]). As far as we know, there is little numerical research on the mITEP. In 2021, [18] studied the spectral Galerkin method for this problem and provided some numerical experiments. In this paper, based on the work of [16], we study the conforming finite element method for the mITEP corresponding to the scattering problem for anisotropic inhomogeneous media. The main work of this paper is as follows:

1. We present a complete error analysis under the assumption of low regularity on the solution. We first derive the a priori error estimates for the source problem associated with the modified interior transmission problem in the metamaterial case (the artificial diffusivity parameter is negative) and the natural case (the artificial diffusivity parameter is positive). To do this, in the metamaterial case we use the Gårding inequality, and in the natural case we use the Gårding inequality and the $\mathbb{T}$-coercivity approach. Then, we prove that the discrete solution operator converges to the exact solution operator and using the spectral approximation theory we obtain the a priori error estimates for the eigenvalue problem.

2. By applying the Gårding inequality and the property of $\mathbb{T}$-coercivity, we derive an a posteriori error formulation (see Lemma 4.1). Then we give the a posteriori error estimator of residual type. Using the technique and method in [19], we prove the reliability and efficiency of the a posteriori error estimator for approximate eigenfunctions, and we also analyze the reliability of estimator for eigenvalues.

3. We present some numerical experiments by using conforming $P1$ and $P2$ elements in adaptive meshes and uniform meshes. Numerical results show that our posteriori error estimator is effective and the approximations can reach the optimal convergence order. It also can be observed from numerical results that conforming finite element eigenvalues approximate the exact values from below, and there exists the monotonic relationship between the conforming finite elements eigenvalues and the refractive index which is also valid for the approximate eigenvalues obtained by the spectral method (see [18]).

The organization of this paper is as follows. In the next section, we introduce the modified transmission eigenvalue problem and its conforming finite element approximation. In Sections 3 and 4, we discuss the a priori and the a posteriori error estimates, respectively. Finally, we present some numerical experiments to validate our theoretical results.