Analysis in Theory and Applications
Anal. Theory Appl., Vol. **39**, No. 3 (2023), pp. 299-308
DOI: 10.4208/ata.OA-2018-0020
September 2023

Null-Controllability of a Diffusion Equation with Fractional Integro-Differential Expressions

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Received 12 June 2018; Accepted (in revised version) 29 September 2019

Abstract. The article considers the controllability of a diffusion equation with fractional integro-differential expressions. We prove that the resulting equation is null-controllable in arbitrary small time. Our method reduces essentially to the study of classical moment problems.

Key Words: Null-controllability, Mittag-Leffler functions, Paley-Wiener type theorems, diffusion equation with fractional integro-differential expressions.

AMS Subject Classifications: 34A08, 30E05, 35E20, 41A75

1 Introduction

Let $f(x)$ be an arbitrary function of class $L_1(0,\sigma)$, $(0<\sigma<\infty)$. Then the Riemann-Liouville fractional integral of $f(x)$ with the origin at the point $x=0$ is taken to be the function

$$D^{-\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt, \quad x\in(0,\sigma).$$

The corresponding fractional derivative of α is defined as follows

$$D^{\alpha}f(x)=\frac{d^p}{dx^p}(D^{-(p-\alpha)}f(x)),$$

where p is an integer satisfying $p-1\le\alpha\le p$.

Let us fix $T,\ell>0$ and let $\{\gamma_j\}_{j=0}^{3}$ be a set of parameters such that

$$1/2<\gamma_0,\gamma_3\le 1, \quad 0\le\gamma_1,\gamma_2\le 1, \quad \sum_{j=0}^{3}\gamma_j=3. \tag{1.1}$$

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All fractional integrals and derivatives are of Riemann-Liouville type. The fractional integrals of f of order $\alpha > 0$, left- and right-handed, respectively, are denoted by ${}_0D_x^{-\alpha} f(x)$ and ${}_xD_\ell^{-\alpha} f(x)$ respectively for $0 < x < \ell$. Define the set of fractional integro-differential expressions associated with the aforementioned parameters (see pp. 178–191 of [1])

$$L_0 f(x) = {}_0 D_x^{1-\gamma_0} f(x), \quad L_j f(x) = {}_0 D_x^{1-\gamma_j} \Big(\frac{d}{dx} L_{j-1} f(x) \Big), \quad j = 1,2,3,$$

and we also define the fractional integro-differential expression by

$$\Lambda_{1/2} f(x) = {}_0 D_x^{-(1-\gamma_3)} \, {}_0D_x^{\gamma_1} \, {}_0D_x^{\gamma_2} \, {}_0D_x^{\gamma_0} f(x).$$

We consider the following diffusion equation with fractional integro-differential expressions

$$\frac{\partial u(x,t)}{\partial t} - \Lambda_{1/2} u(x,t) = a(x) v(t), \tag{1.2}$$

$x \in (0,\ell)$, $t \in (0,T)$, with the boundary conditions

$$L_m u(x,t) = 0 \tag{1.3}$$

for $m = 0,1,2$, where $a(x) \in L^2(0,\ell)$, $\{\gamma_j\}_{j=0}^3$ is a set of parameters defined in (1.1) and

$${}_0D^{2-\gamma_2-\gamma_3} D^{\gamma_0} u(\ell,t) = 0, \quad u(x,0) = u_0, \quad u(\ell,t) = 0. \tag{1.4}$$

The function v in the above equation is called a control.

The problem which we consider for this fractional order diffusion equation is connected to the null-controllability, by which we mean that, for any given initial state u_0, find (if possible) a control v such that the solution of (1.2) satisfies $u(x,T) = 0$ for all $x \in (0,\ell)$. It is well known that the method and theory of fractional differential equations have many applications in various fields of engineering and science, for example, vibration, control and electromagnetic theory. We believe that the study of null-controllability of fractional differential equations is a worthwhile endeavor. There are many recent works which are devoted to study in this filed. For example, in [8], S. Micu and E. Zuazua studied the null-controllability property of a parabolic equation involving a fractional power of the Laplace operator, i.e., $(-\Delta)^\alpha$; in [9], S. Micu, J. H. Ortega and A. F. Pazoto considered a diffusion equation perturbed by a vanishing viscosity term, they characterized the null-controllability property of the diffusion equation involving a fractional power of the Laplace operator, i.e., $(-\Delta)^{1/2}$; K. Balachandran, Y. Zhou and J. Kokila investigated controllability for nonlinear dynamical systems of fractional order with delays in [2]; the controllability for a class of fractional integro-differential evolution equations with nonlocal initial conditions was also considered by J. Liang and H.Yang in the paper [6]. In [12], the Paley-Wiener type theorem for Mittag-Leffer type functions were used to study the controllability of a fractional order diffusion equation.