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ON DISCRETE ENERGY DISSIPATION OF MAXWELL'S EQUATIONS IN A COLE-COLE DISPERSIVE MEDIUM*

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Abstract

A simple criterion is studied for the first time for identifying the discrete energy dissipation of the Crank-Nicolson scheme for Maxwell's equations in a Cole-Cole dispersive medium. Several numerical formulas that approximate the time fractional derivatives are investigated based on this criterion, including the L1 formula, the fractional BDF-2, and the shifted fractional trapezoidal rule (SFTR). Detailed error analysis is provided within the framework of time domain mixed finite element methods for smooth solutions. The convergence results and discrete energy dissipation law are confirmed by numerical tests. For nonsmooth solutions, the method SFTR can still maintain the optimal convergence order at a positive time on uniform meshes. Authors believe this is the first appearance that a second-order time-stepping method can restore the optimal convergence rate for Maxwell's equations in a Cole-Cole dispersive medium regardless of the initial singularity of the solution.

Mathematics subject classification: 65N06, 65B99.
Key words: Discrete energy dissipation, Crank-Nicolson scheme, Maxwell's equations, Shifted fractional trapezoidal rule, Mixed finite element methods.

1. Introduction

Since the early 1990's, studies on wave propagation in medium such as water, soil, biological tissue, ionosphere, plasma, optical fiber and radar absorbing material, etc. [3, 17, 25, 29] have aroused interest of engineers for the common property that the medium's permittivity or permeability depends on the wave frequency. Models investigating this dependency include the Drude model [46], the Lorenz model [32] and the anomalously dispersive model such as the Havriliak-Negami model [9, 10], Cole-Cole model [6], and so forth. Numerical techniques for these models cover the finite difference time-domain (FDTD) methods [4,28,31,36,37], finite element time-domain (FETD) methods [2,12,15,17], spectral time-domain (STD) methods [11,39] and discontinuous Galerkin time-domain (DGTD) methods [21, 38], among others. Attempts in frequency domain can also be found in [1, 5, 23, 26, 27] and references cited therein.

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In this study, we discuss the discrete energy dissipation law of the 2-D Maxwell's equations in a Cole-Cole dispersive medium [17], which can be stated as

$$\epsilon_0\epsilon_\infty\frac{\partial \boldsymbol{E}}{\partial t} = \nabla\times H - \frac{\partial \boldsymbol{P}}{\partial t}, \tag{1.1}$$

$$\mu_0\frac{\partial H}{\partial t} = -\nabla\times \boldsymbol{E}, \tag{1.2}$$

$$\tau_0^\alpha\partial_t^\alpha \boldsymbol{P}(t) + \boldsymbol{P}(t) = \epsilon_0(\epsilon_s-\epsilon_\infty)\boldsymbol{E}(t), \tag{1.3}$$

where $\boldsymbol{E}(\boldsymbol{x},t) = (E_1,E_2)^T, H(\boldsymbol{x},t)$ with $\boldsymbol{x} = (x_1,x_2)^T \in \Omega = (a,b)\times(c,d)$, $t\in(0,T]$, represent the electric field and magnetic field, respectively. $\boldsymbol{P}(\boldsymbol{x},t)$ is the polarization field in time-domain. $\epsilon_0,\epsilon_\infty$ and ϵ_s denote respectively the permittivity in the free space, infinite-frequency permittivity and the static permittivity with the relation $\epsilon_s > \epsilon_\infty$. Besides, μ_0 is the permeability of free space and τ_0 is the relaxation time. Here we define

$$\nabla\times H = \left(\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x}\right)^T, \quad \nabla\times\boldsymbol{E} = \frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}.$$

To close the problem (1.1)-(1.3), the following initial-boundary conditions are needed [17]:

$$\boldsymbol{E}(\boldsymbol{x},0) = \boldsymbol{E}_0(\boldsymbol{x}), \quad H(\boldsymbol{x},0) = H_0(\boldsymbol{x}), \quad \boldsymbol{P}(\boldsymbol{x},0) = 0 \quad \text{for} \quad \boldsymbol{x}\in\overline{\Omega}, \tag{1.4}$$

and

$$\boldsymbol{n}\times\boldsymbol{E} = 0 \quad \text{on} \ \partial\Omega\times[0,T]. \tag{1.5}$$

The fractional derivative operator ∂_t^α defined by

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u_s(s)\mathrm{d}s}{(t-s)^\alpha}, \tag{1.6}$$

is known as the Caputo fractional derivative. In this work, we shall take the strategy to approximate the auxiliary differential equation (ADE) to deal with the polarisation for its straightforward and easy implementation. Thus, the fractional derivative will be approximated directly by some time-stepping methods of the form

$$\partial_t^{\alpha,n-\frac12}u = \partial_t^\alpha u(t_{n-\frac12}) \approx \partial_\tau^{\alpha,n-\frac12}u := \tau^{-\alpha}\sum_{j=0}^n \omega_j u^{n-j}, \tag{1.7}$$

where τ denotes the time mesh size, and ω_j are weights depending on different approximation formulas.

It is shown in [17] that the model (1.1)–(1.3) admits the following energy dissipation law:

$$\begin{aligned}&\mathcal{E}(t)\le\mathcal{E}(0) \quad \text{for any} \quad t\in[0,T],\\ &\mathcal{E}(t) = \epsilon_0(\epsilon_s-\epsilon_\infty)\big(\epsilon_0\epsilon_\infty\|\boldsymbol{E}(t)\|_0^2 + \mu_0\|H(t)\|_0^2\big) + \|\boldsymbol{P}(t)\|_0^2.\end{aligned} \tag{1.8}$$

However, the discrete version of (1.8) is still unknown to the best of our knowledge, which motivates us to propose a criterion for the discrete energy dissipation when the Crank-Nicolson scheme is adopted in temporal direction and fractional derivatives are approximated by formulas like (1.7). We would like to mention that the study on the energy dissipation of the problem (1.1)–(1.3) is rather limited. Yang *et al.* [39] studied the Havriliak-Negami model which is a generalization of the Cole-Cole model by appealing to the approximation of the induced