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AN ADAPTIVE FINITE ELEMENT METHOD FOR SHAPE OPTIMIZATION IN STATIONARY INCOMPRESSIBLE FLOW WITH DAMPING

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Abstract. This paper develops an adaptive finite element method for shape optimization in stationary incompressible flow with damping. The continuous shape gradient of an objective functional with respect to the boundary shape is derived by using the adjoint equation method and a function space parametrization technique. A projection a-posteriori error estimator is proposed, which can be computed easily and implemented in parallel. Based on this error estimator, an adaptive finite element method is constructed to solve state and adjoint equations and a regularized equation in each iteration step. Finally, the effectiveness of this adaptive method is demonstrated by numerical experiments.

Key words. Shape gradient, a-posteriori error estimator, projection operator, adaptive finite element method, numerical experiments.

1. Introduction

A shape optimization problem is to find a domain in a set of admissible domains such that an objective functional achieves a minimum or maximum on it [22]. The research of shape optimization is a branch of optimal control governed by partial differential equations [15] and has a very wide range of applications in engineering such as in the design of aircraft wings, high-speed train heads, impeller blades, and bridges in medically bypassing surgeries. In the last few decades, the shape optimization problems have attracted the interest of many applied mathematicians and engineers [11-14, 16-18, 22, 23, 26, 27].

Numerical methods for shape optimization problems can be classified into gradient-based and non-gradient-based optimization methods. The non-gradient-based methods include the one-shot method [11], approximate model methods [13, 18], and evolutionary methods [16, 17]. The one-shot method does not involve an optimization iteration and only needs to solve an optimality system which consists of coupled state and adjoint equations and an optimality condition. The one-shot method seems very attractive but it is not feasible to solve a coupled large-scale nonlinear system in many flow optimization and control problems [11]. The approximate model methods such as the response surface method and the Kriging method depend on the choice of a sample space. If the early samples cannot reflect the characteristics of the design space, these methods will fail to find an optimal shape. The evolutionary methods may be able to find a global minimum or maximum when the strained state equations are easy to solve. However, these methods are difficult to use in reality when a cost function in them is difficult to calculate, because they involve hundreds or even thousands of calculations to locate a near-optimal solution even for fairly simple cases.

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Relatively, the gradient-based methods have the advantage of fast convergence and high efficiency. For these methods, the most crucial step is how to compute the gradient of an objective functional with respect to a shape variable. The approaches to obtain the shape gradient include the finite difference [11], sensitivity [12, 22] and adjoint equation approaches [14, 26, 27, 30]. The finite difference approach finds a gradient by using a difference quotient approximation. Thus, if N design variables are used to describe a domain shape, then one needs to solve the constrained state equations $N+1$ times at each iteration step of the optimization algorithm. This approach can be extremely expensive in practical applications involving a large number of design variables. The sensitivity approach utilizes a sensitivity equation to obtain the shape gradient by the chain rule, and only requires to solve the state equations one time and linear sensitivity systems N times at each optimization cycle. In contrast, to compute all components of the gradient of the functional using the adjoint equation approach requires the solutions of a single linear adjoint equation and state equations one time. This approach produces a gradient of the objective functional without a cost increase with an increasing number of shape design parameters.

In every optimal cycle, how to increase the accuracy of numerical approximations for a shape gradient is still a big challenge. The overall accuracy of the numerical approximations often deteriorates due to local singularities such as those arising from corners of domains and interior or boundary layers. An obvious strategy is to refine the grids near these critical regions, i.e., to insert more grid points where the singularities occur. A mathematical theory is developed for an adaptive finite element method based on a class of a-posterior error estimators by Babuška and Rheinboldt [1]. Yan and Liu et al derived a-posteriori error estimates for a finite element approximation of distributed optimal control problems governed by the Stokes equations [2] and parabolic equations[31]. Bangerth introduced a framework for the adaptive finite element solution of a coefficient estimation problem in partial differential equations [3]. In 2010, Zee et al. developed duality-based a-posteriori error estimates and adaptivity for free boundary problems via shape-linearization principles [4].

In this paper, we study an adaptive finite element method for shape optimization in stationary incompressible flow with damping. First, we use a velocity method to describe a variational domain in the optimization process. Second, the adjoint equations are derived by employing the differentiability of an saddle point problem which includes a Lagrange multiplier function. We obtain the continuous gradient of an objective functional with respect to the domain shape with these adjoint equations and a function space parametrization technique. Third, motivated by the stabilized finite element method based on the two local Gauss integrals technique in recent years [19-21, 24, 25], we construct an a-posterior error estimator by a projection operator. Fourth, we present the adaptive finite element method for the state and adjoint equations and a regularized gradient equation based on this error estimator. Finally, the effectiveness of this adaptive method is demonstrated by numerical experiments.

This paper is organized as follows: In Section 2, we state the shape optimization problem in stationary incompressible flow with damping and derive the continuous shape gradient. In Section 3 we propose an adaptive finite element method based on a projection a-posteriori error estimator. We then present numerical examples in Section 4, followed by conclusions in Section 5.