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PI-VEGAN: Physics Informed Variational Embedding Generative Adversarial Networks for Stochastic Differential Equations

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Abstract. We present a new category of physics-informed neural networks called physics informed variational embedding generative adversarial network (PI-VEGAN), that effectively tackles the forward, inverse, and mixed problems of stochastic differential equations. In these scenarios, the governing equations are known, but only a limited number of sensor measurements of the system parameters are available. We integrate the governing physical laws into PI-VEGAN with automatic differentiation, while introducing a variational encoder for approximating the latent variables of the actual distribution of the measurements. These latent variables are integrated into the generator to facilitate accurate learning of the characteristics of the stochastic partial equations. Our model consists of three components, namely the encoder, generator, and discriminator, each of which is updated alternatively employing the stochastic gradient descent algorithm. We evaluate the effectiveness of PI-VEGAN in addressing forward, inverse, and mixed problems that require the concurrent calculation of system parameters and solutions. Numerical results demonstrate that the proposed method achieves satisfactory stability and accuracy in comparison with the previous physics-informed generative adversarial network (PI-WGAN).

AMS subject classifications: 60H35, 34F05, 62M45

Key words: Stochastic differential equations, physics-informed, variational inference, generative adversarial networks, inverse problems.

1. Introduction

Stochastic differential equations (SDEs) arise in many fields, including finance, physics, and engineering, and typically involve random fluctuations in the underlying system. The computational methods used to solve these equations must be able to

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handle both the spatial and temporal variability of the problem, as well as the stochastic nature of the solution.

Over the past four decades, there has been a significant increase in the development of numerical methods for stochastic differential equations. Monte Carlo methods, which involve simulating the underlying stochastic process using random sampling techniques, are commonly used to solve SDEs. Other numerical methods for SDEs include spectral methods, which use Fourier or wavelet transforms to solve the equation in the frequency domain, and stochastic Galerkin methods, which involve projecting the SDE onto a finite-dimensional space and solving the resulting system of equations using standard numerical techniques. For instance, Kloeden and Platen [21] examined various strong and weak approximation methods based on the stochastic Taylor formula for Ito stochastic differential equations. Stability and implementation issues have been discussed in [4], with variable step size implementation shown to be superior to fixed step size for small stochasticity. Tocino [25] developed a class of explicit Runge-Kutta schemes of second order in the weak sense for systems of stochastic differential equations with multiplicative noise. Furthermore, two Runge-Kutta schemes of third order were obtained for scalar equations with constant diffusion coefficients, where the first method generated independent identically distributed approximations of the solution by sampling the coefficients of the equation and used a standard Galerkin finite element variational formulation. Babuska *et al.* [1] proposed a finite dimensional approximation of the stochastic coefficients, turning the original stochastic problem into a deterministic parametric elliptic problem. A Galerkin finite element method of either the h- or p-version was then used to approximate the corresponding deterministic solution, which led to approximations of the desired statistics. However, traditional numerical methods may not be efficient for high dimensional stochastic partial equations and can suffer from the curse of dimensionality.

In recent years, the use of deep learning to solve fundamental partial differential equations (PDEs) has gained considerable attention [10, 24, 26], thanks to the high expressiveness of neural networks and the rapid growth of computing hardware. Among them, physics-informed neural networks (PINNs) [5, 11, 16, 17, 19, 23, 27–29, 31] are a particularly interesting approach. PINNs incorporate physical knowledge as soft constraints in the empirical loss function and employ machine learning methodologies like automatic differentiation and stochastic optimization to train the model. In [22], the random PDE is approximated by a feed-forward deep residual network, with either strong or weak enforcement of initial and boundary constraints. A reinforcement learning method was presented in [9] to solve backward stochastic differential equations, where the gradient of the solution plays the role of a policy function and the loss function is given by the error between the prescribed terminal condition and the solution. Chen *et al.* [6] employed a Karhunen-Loève expansion for the stochastic diffusivity and arbitrary polynomial chaos for the solution, and then designed multiple neural networks to solve stochastic advection–diffusion–reaction systems. A machine learning method, lifting the requirement for a deterministic forward solver, was presented by [18] for high-dimensional uncertainty propagation of elliptic SDEs. In many practi-