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Novel High-Order Mass- and Energy-Conservative Runge-Kutta Integrators for the Regularized Logarithmic Schrödinger Equation

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Abstract. We develop a class of conservative integrators for the regularized logarithmic Schrödinger equation (RLogSE) using the quadratization technique and symplectic Runge-Kutta schemes. To preserve the highly nonlinear energy functional, the regularized equation is first transformed into an equivalent system that admits two quadratic invariants by adopting the invariant energy quadratization approach. The reformulation is then discretized using the Fourier pseudo-spectral method in the space direction, and integrated in the time direction by a class of diagonally implicit Runge-Kutta schemes that conserve both quadratic invariants to round-off errors. For comparison purposes, a class of multi-symplectic integrators are developed for RLogSE to conserve the multi-symplectic conservation law and global mass conservation law in the discrete level. Numerical experiments illustrate the convergence, efficiency, and conservative properties of the proposed methods.

AMS subject classifications: 35Q55, 65L06, 81Q05

Key words: Regularized logarithmic Schrödinger equation, conservative numerical integrators, invariant energy quadratization approach, diagonally implicit Runge-Kutta scheme.

1. Introduction

In this paper, we are concerned with conservative numerical integrations of a specific type of nonlinear Schrödinger equations (NLSEs), i.e., the regularized version of the logarithmic Schrödinger equation (LogSE). The LogSE which was introduced as a model of nonlinear wave mechanics [7] as follows:

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$$\begin{cases} \mathrm{i}u_t(\mathbf{x},t)+\Delta u(\mathbf{x},t)=\lambda u(\mathbf{x},t)\ln\left(|u(\mathbf{x},t)|^2\right), & \mathbf{x}\in\Omega, \quad t>0, \\ u(\mathbf{x},0)=u_0(\mathbf{x}), & \mathbf{x}\in\bar{\Omega} \end{cases} \tag{1.1}$$

with homogeneous Dirichlet boundary conditions or periodic boundary conditions posted on the boundary. It has found wide applications in different branches of fundamental physics, such as nuclear physics [22], quantum optics [8], diffusion phenomena [21], and Bose-Einstein condensation [52].

Similar to the usual NLSE with power-like nonlinearity,

$$\mathrm{i}u_t(\mathbf{x},t)+\Delta u(\mathbf{x},t)=\mu|u(\mathbf{x},t)|^{2\sigma}u(\mathbf{x},\mathbf{t}), \quad \mathbf{x}\in\Omega, \quad t>0, \tag{1.2}$$

the LogSE (1.1) conserves the global mass, momentum, and energy [12], which are defined, respectively, as

$$M(u):=\int_\Omega |u|^2\mathrm{d}\mathbf{x}\equiv M(u_0), \tag{1.3}$$

$$P(u):=\mathrm{Im}\int_\Omega \overline{u}\nabla u\mathrm{d}\mathbf{x}\equiv P(u_0), \tag{1.4}$$

$$E(u):=\int_\Omega \left(|\nabla u|^2+\lambda|u|^2\ln(|u|^2)\right)\mathrm{d}\mathbf{x}\equiv E(u_0). \tag{1.5}$$

In the LogSE, the function $u\to u\ln(|u|^2)$ is not Lipschitz continuous at $u=0$ because of the singularity of the logarithm at the origin; thus, one cannot directly apply the schemes developed for NLSE to LogSE. To avoid numerical blow-ups and suppress round-off errors due to the logarithmic nonlinearity in the LogSE, a regularized logarithmic Schrödinger equation (RLogSE) with a small parameter $0<\epsilon\ll 1$ was introduced by Bao *et al.* [2] as

$$\begin{cases} \mathrm{i}u^\epsilon_t(\mathbf{x},t)+\Delta u^\epsilon(\mathbf{x},t)=\lambda u^\epsilon(\mathbf{x},t)\ln\left(\epsilon+|u^\epsilon(\mathbf{x},t)|\right)^2, & \mathbf{x}\in\Omega, \quad t>0, \\ u^\epsilon(\mathbf{x},0)=u_0(\mathbf{x}), & \mathbf{x}\in\bar{\Omega}. \end{cases} \tag{1.6}$$

The RLogSE (1.6) approximates the LogSE (1.1) with linear convergence rate $\mathcal{O}(\epsilon)$, and has conservation laws similar to those of the original model, i.e., conservation of mass $M^\epsilon(u^\epsilon):=M(u^\epsilon)$ and momentum $P^\epsilon(u^\epsilon):=P(u^\epsilon)$, as well as the regularized energy conservation law [2] defined as

$$E^\epsilon(u^\epsilon):=\int_\Omega\left[|\nabla u^\epsilon|^2+2\epsilon\lambda|u^\epsilon|+2\lambda(|u^\epsilon|^2-\epsilon^2)\ln(\epsilon+|u^\epsilon|)\right]\mathrm{d}\mathbf{x}\equiv E^\epsilon(u_0).$$

Denoting $\rho(u^\epsilon)=|u^\epsilon|^2$, it holds that

$$\begin{aligned} E^\epsilon(u^\epsilon)&=\int_\Omega\left[|\nabla u^\epsilon|^2+\lambda F_\epsilon(\rho(u^\epsilon))+\lambda\rho(u^\epsilon)\right]\mathrm{d}\mathbf{x}, \\ F_\epsilon(\rho)&=\int_0^\rho f_\epsilon(s)\mathrm{d}s=2(\rho-\epsilon^2)\ln(\epsilon+\sqrt{\rho})-\rho+2\epsilon\sqrt{\rho}, \\ f_\epsilon(\rho)&=\frac{\mathrm{d}F_\epsilon(\rho)}{\mathrm{d}\rho}=2\ln(\epsilon+\sqrt{\rho}). \end{aligned}$$