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A CONFORMING DG METHOD FOR THE BIHARMONIC EQUATION ON POLYTOPAL MESHES

XIU YE AND SHANGYOU ZHANG

Abstract. A conforming discontinuous Galerkin finite element method is introduced for solving the biharmonic equation. This method, by its name, uses discontinuous approximations and keeps simple formulation of the conforming finite element method at the same time. The ultra simple formulation of the method will reduce programming complexity in practice. Optimal order error estimates in a discrete H^2 norm is established for the corresponding finite element solutions. Error estimates in the L^2 norm are also derived with a sub-optimal order of convergence for the lowest order element and an optimal order of convergence for all high order of elements. Numerical results are presented to confirm the theory of convergence.

Key words. finite element methods, weak Laplacian, biharmonic equations, polyhedral meshes.

1. Introduction

We consider the biharmonic equation of the form

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{3}$$

where Ω is a bounded polytopal domain in $\mathbb{R}^d$.

The weak formulation of the boundary value problem (2) and (3) is seeking $u \in H_0^2(\Omega)$ satisfying

$$(\Delta u, \Delta v) = (f, v) \qquad \forall v \in H_0^2(\Omega). \tag{4}$$

The H^2 conforming finite element method for the problem (1)-(3) keeps the same simple form as in (4): find $u_h \in V_h \subset H_0^2(\Omega)$ such that

$$(\Delta u_h, \Delta v) = (f, v) \quad \forall v \in V_h. \tag{5}$$

The early works in the area of finite elements (in 1960s) are mostly constructions of conforming and nonconforming elements for solving the biharmonic equation, for example, the Argyris element (1968), the Bell element (1969), the Bogner-Fox-Schmit rectangle (1965), the Hsieh-Clough-Tocher element (1965), the Fraeijs de Veubeke-Sander (1964) and the Morley element (1969), cf. [1, 2, 3, 11, 16, 44, 34]. Many more publications on C^1-conforming and nonconforming finite elements can be found in [12, 14, 19, 22, 24, 25, 26, 27, 28, 29, 31, 33, 40, 41, 42, 47, 48, 56, 57, 58, 59, 60, 61, 62]. Some alternative methods are the interior penalty discontinuous Galerkin finite element methods [13, 17, 18, 35, 36, 46], the mixed finite elements of two Laplacians [4, 8, 9, 10, 15, 32, 43, 45], the Hellan-Herrmann-Johnson element [5, 20, 21, 30], and $H(\text{div}\mathbf{div})$ mixed finite elements [6, 7, 23, 55].

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An approach of avoiding construction of H^2-conforming elements is to use discontinuous approximations. Due to the flexibility of discontinuous Galerkin (DG) finite element methods in element constructions and in mesh generations, many finite element methods have been developed using totally discontinuous polynomials. Here we are only interested in interior penalty discontinuous Galerkin (IPDG) methods since the proposed the method shares the same finite element spaces with IPDG method [13, 17, 18, 35, 36, 46]. One obvious disadvantage of discontinuous finite element methods is their rather complicated formulations which are often necessary to guarantee well posedness and convergence of the methods. For example, the symmetric IPDG method for the biharmonic equation with homogenous boundary conditions [13, 17] has the following formulation:

$$
\begin{aligned}
(\Delta u_h, \Delta v)_{\mathcal{T}_h} \quad &+ \quad \sum \int_e (\{\nabla \Delta u_h\} \cdot [v] + \{\nabla \Delta v\} \cdot [u_h]) ds \\
&+ \quad \sum \int_e (\{\Delta u_h\} \cdot [\nabla v] + \{\Delta v\} \cdot [\nabla u_h]) ds \\
&+ \quad \sum \int_e (\sigma [u_h] \cdot [v] + \tau [\nabla u_h][\nabla v]) ds = (f, v), \qquad (6)
\end{aligned}
$$

where σ and τ are two parameters that need to be tuned.

The purpose of this work is to introduce a conforming DG finite element method for the biharmonic equation which has the following ultra simple formulation without any stabilizing/penalty terms and other mixed terms of lower dimension integrals in (6):

$$
(\Delta_w u_h,\ \Delta_w v) = (f,\ v), \qquad (7)
$$

where Δ_w is called weak Laplacian, an approximation of Δ. The formulation (7) can be viewed as a counterpart of (5) for discontinuous approximations. The conforming DG method was first introduced in [52, 53] for second order elliptic equations. A conforming DG method, by name, means the method using the simple formulation of the conforming finite element method and the spaces of some DG finite element methods. That is, in the finite element equations, there is no penalty term neither any consistence-error control term.

This conforming DG finite element method (7) shares the same finite element space with the IPDG methods but having much simpler formulation. This simple formulation is obtained by defining weak Laplacian Δ_w appropriately. The idea here is to raise the degree of polynomials used to compute the weak Laplacian Δ_w. Using higher degree polynomials in computing the weak Laplacian will not change the size, neither the global sparsity of the stiffness matrix. Optimal order error estimates in a discrete H^2 for $k \geq 2$ and in L^2 norm for $k > 2$ are established for the corresponding finite element solutions. Numerical results are provided, confirming the theory.

The analysis of conforming DG finite element methods is based on that of the weak Galerkin finite element methods. [37] is the first weak Galerkin finite element work on the biharmonic equation. [39] is on the weak Galerkin method with the C^0 finite element spaces, for solving the biharmonic equation. A stabilizer free weak Galerkin method for the biharmonic equation is designed in [54]. This work is based on [54], by eliminating further the auxiliary variables of the function and the normal derivative on the inter-element edges or polygons.