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Gevrey Well-Posedness of Quasi-Linear Hyperbolic Prandtl Equations

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Abstract. We study the hyperbolic version of the Prandtl system derived from the hyperbolic Navier-Stokes system with no-slip boundary condition. Compared to the classical Prandtl system, the quasi-linear terms in the hyperbolic Prandtl equation leads to an additional instability mechanism. To overcome the loss of derivatives in all directions in the quasi-linear term, we introduce a new auxiliary function for the well-posedness of the system in an anisotropic Gevrey space which is Gevrey class 3/2 in the tangential variable and is analytic in the normal variable.

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Key words: Hyperbolic Prandtl equations, quasi-linear, Gevrey class.

1 Introduction

We investigate the well-podedness of the following quasi-linear hyperbolic Prandtl system in the half-space $\mathbb{R}^d_+ \stackrel{\text{def}}{=} \{(x,y);\ x\in\mathbb{R}^{d-1}, y>0\}$ with $d=2$ or 3:

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$$\begin{cases} \eta\partial_t^2 u+\partial_t u+(u\cdot\partial_x)u+v\partial_y u+\eta\partial_t\big((u\cdot\partial_x)u+v\partial_y u\big)-\partial_y^2 u+\partial_x p=0, \\ \partial_x u+\partial_y v=0, \\ u|_{y=0}=v|_{y=0}=0, \quad u|_{y\to+\infty}=U, \\ u|_{t=0}=u_0, \qquad\qquad \partial_t u|_{t=0}=u_1, \end{cases} \tag{1.1}$$

where $0<\eta<1$ is a small parameter. The unknown u represents the tangential velocity which is scalar in the two-dimensional (2D) case and vector-valued in 3D. And the functions $p=p(t,x)$ and $U=U(t,x)$ in (1.1) are the traces of the tangential velocity field and pressure of the outer flow on the boundary satisfying that

$$\eta\partial_t^2 U+\partial_t U+U\cdot\partial_x U+\eta\partial_t(U\cdot\partial_x U)+\partial_x p=0.$$

This degenerate hyperbolic system (1.1) can be derived from the hyperbolic Navier-Stokes equations with the no-slip boundary condition. It is well-known that the classical Navier-Stokes system can be obtained from the Newtonian law. And its parabolic structure leads to the property of infinite speed of propagation which seems to be a paradox from the physical point of view. To have finite propagation, Cattaneo [4,5] proposed to replace the Fourier law by the so-called Cattaneo law, where a small time delay η is introduced in stress tensors. And this yields the following hyperbolic version of Navier-Stokes equations:

$$\eta\partial_t^2 u^{NS}+\partial_t u^{NS}+(u^{NS}\cdot\nabla)u^{NS}+\eta\partial_t\big((u^{NS}\cdot\nabla)u^{NS}\big)-\varepsilon\Delta u^{NS}+\nabla p^{NS}=0, \tag{1.2}$$

where the gradient operator ∇ is taken with respect to all spatial variables, and similarly for the Laplace operator Δ. In the whole space, the system (1.2) with fixed viscosity $\varepsilon>0$ was studied by Coulaud *et al.* [7] in almost optimal function spaces (see also [2,30]). On the other hand, it is natural to study the inviscid limit of (1.2) as $\varepsilon\to 0$, in particular in the situation when the fluid domain has a physical boundary. In fact, when we analyze the asymptotic expansion with respect to the viscosity ε of (1.2) with the no-slip boundary condition, a Prandtl type boundary layer is expected to take care of the mismatched tangential velocities. In fact, the governing equation of the boundary layer is the system (1.1) by following the Prandtl's ansatz.

When $\eta=0$, the system (1.1) is the classical Prandtl equations. The mathematical study of the classical Prandtl boundary layer has a long history with fruitful results and developed approaches in analysis. It has been well studied in various function spaces, see, e.g. [3,6,8–10,12,14–16,18,20,23,25–28,32–35] and the references therein. Due to the loss of tangential derivatives in the nonlocal term $v\partial_y u$, the Prandtl system is usually ill-posed in Sobolev spaces. It is now well understood that, for initial data without any structural assumption, the Prandtl system