Commun. Math. Anal. Appl.
doi: 10.4208/cmaa.2023-0008

Vol. **2**, No. 4, pp. 421-468
November 2023

Global Regularity of the Vlasov-Poisson-Boltzmann System Near Maxwellian Without Angular Cutoff for Soft Potential

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Received 14 September 2023; Accepted 8 October 2023

Abstract. We consider the non-cutoff Vlasov-Poisson-Boltzmann (VPB) system of two species with soft potential in the whole space $\mathbb{R}^3$ when an initial data is near Maxwellian. Continuing the work Deng [Comm. Math. Phys. 387 (2021)] for hard potential case, we prove the global regularity of the Cauchy problem to VPB system for the case of soft potential in the whole space for the whole range $0<s<1$. This completes the smoothing effect of the Vlasov-Poisson-Boltzmann system, which shows that any classical solutions are smooth with respect to (t,x,v) for any positive time $t>0$. The proof is based on the time-weighted energy method building upon the pseudo-differential calculus.

AMS subject classifications: 76X05, 35Q20, 76P05, 82C40

Key words: Vlasov-Poisson-Boltzmann system, regularity, without angular cutoff, regularizing effect, soft potentials.

1 Introduction

The Vlasov-Poisson-Boltzmann system is an important physical model to describe the time evolution of plasma particles of two species (e.g. ions and electrons). In this work, we study the smoothing effect of solutions to non-cutoff

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Vlasov-Poisson-Boltzmann system with $-3/2-2s<\gamma\le -2s$ and $0<s<1$. We find that the solutions enjoy the same smoothing phenomenon as the Boltzmann equation, which gives the regularity of the Vlasov-Poisson-Boltzmann system. Since Duan and Liu [17] found the global solution for non-cutoff soft potential with $1/2\le s<1$, the smoothing effect for the VPB system is an open interesting problem. In [14], the author finds out the smoothing effect for hard potential. In this work, we finally recover the smoothing effect for non-cutoff soft potential with the whole range $0<s<1$.

1.1 Equations

We consider the Vlasov-Poisson-Boltzmann system of two species in the whole space $\mathbb{R}^3$, cf. [22,26]

$$\begin{aligned} &\partial_t F_+ + v\cdot\nabla_x F_+ + E\cdot\nabla_v F_+ = Q(F_+,F_+)+Q(F_-,F_+),\\ &\partial_t F_- + v\cdot\nabla_x F_- - E\cdot\nabla_v F_- = Q(F_-,F_-)+Q(F_+,F_-). \end{aligned} \tag{1.1}$$

The self-consistent electrostatic field is taken as $E(t,x)=-\nabla_x\phi$, with the electric potential ϕ given by

$$-\Delta_x\phi=\int_{\mathbb{R}^3}(F_+-F_-)\,dv,\quad \phi\ \to\ 0\quad \text{as}\quad |x|\to\infty. \tag{1.2}$$

The initial data of the system is

$$F_\pm(0,x,v)=F_{\pm,0}(x,v). \tag{1.3}$$

The unknown function $F_\pm(t,x,v)\ge 0$ represents the velocity distribution for the particle with position $x\in\mathbb{R}^3$ and velocity $v\in\mathbb{R}^3$ at time $t\ge 0$. The bilinear collision term $Q(F,G)$ on the right-hand side of (1.1) is given by

$$Q(F,G)(v)=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\big(F'_*G'-F_*G\big)\,d\sigma dv_*,$$

where

$$F'=F(x,v',t),\quad G'_*=G(x,v'_*,t),\quad F=F(x,v,t),\quad G_*=G(x,v_*,t).$$

Velocity pairs (v,v_*) and (v',v'_*) are velocities before and after binary elastic collision respectively. They are defined by

$$v'=\frac{v+v_*}{2}+\frac{|v-v_*|}{2}\sigma,\quad v'_*=\frac{v+v_*}{2}-\frac{|v-v_*|}{2}\sigma.$$

These two pairs of velocities satisfy the conservation law of momentum and energy

$$v+v_*=v'+v'_*,\quad |v|^2+|v_*|^2=|v'|^2+|v'_*|^2.$$