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UNIFORM ERROR BOUNDS OF A CONSERVATIVE COMPACT FINITE DIFFERENCE METHOD FOR THE QUANTUM ZAKHAROV SYSTEM IN THE SUBSONIC LIMIT REGIME*

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Abstract

In this paper, we consider a uniformly accurate compact finite difference method to solve the quantum Zakharov system (QZS) with a dimensionless parameter $0 < \varepsilon \leq 1$, which is inversely proportional to the acoustic speed. In the subsonic limit regime, i.e., when $0 < \varepsilon \ll 1$, the solution of QZS propagates rapidly oscillatory initial layers in time, and this brings significant difficulties in devising numerical algorithm and establishing their error estimates, especially as $0 < \varepsilon \ll 1$. The solvability, the mass and energy conservation laws of the scheme are also discussed. Based on the cut-off technique and energy method, we rigorously analyze two independent error estimates for the well-prepared and ill-prepared initial data, respectively, which are uniform in both time and space for $\varepsilon \in (0,1]$ and optimal at the fourth order in space. Numerical results are reported to verify the error behavior.

Mathematics subject classification: 35Q55, 65M06, 65M12, 65M15.
Key words: Quantum Zakharov system, Subsonic limit, Compact finite difference method, Uniformly accurate, Error estimate.

1. Introduction

Consider the quantum Zakharov system (QZS) for describing the nonlinear interaction between high-frequency quantum Langmuir and low-frequency quantum ion-acoustic waves [12,17],

$$\begin{cases} iE_t^\varepsilon + \Delta E^\varepsilon - \lambda^2\Delta^2 E^\varepsilon = N^\varepsilon E^\varepsilon, & \\ \varepsilon^2 N_{tt}^\varepsilon - \Delta N^\varepsilon + \lambda^2 \Delta^2 N^\varepsilon = \Delta |E^\varepsilon|^2, & x \in \mathbb{R}^d, \quad t > 0, \\ E^\varepsilon(x,0) = E_0(x), \quad N^\varepsilon(x,0) = N_0^\varepsilon(x), \quad \partial_t N^\varepsilon(x,0) = N_1^\varepsilon(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $i^2 = -1$, $E^\varepsilon : \mathbb{R}^{d+1} \to \mathbb{C}$ denotes the slowly varying envelope of the rapidly oscillatory electric field, $N^\varepsilon : \mathbb{R}^{d+1} \to \mathbb{R}$ represents the low-frequency variation of the density of the ions. The dimensionless parameter $\varepsilon \in (0,1]$ is inversely proportional to the speed of ion sound, the quantum effect $\lambda > 0$ is the ratio of the ion plasma and the temperature of electrons, and

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$E_0(x)$, $N_0^\varepsilon(x)$, and $N_1^\varepsilon(x)$ are given initial datum. The QZS are deduced from a multiple time-scale method applied to a set of quantum hydrodynamic (QHD) equations under quasineutral assumption [17]. It extends the classical Zakharov system ($\lambda = 0$) [31] to the quantum realm. When either the ion-plasma frequency is high or the electrons temperature is low, the quantum effect is non-negligible and can be characterized by the fourth-order perturbation with a quantum parameter λ. It can be applied for quantum decay and four-wave instabilities with relevant changes of the classical dispersion [12], or for the enhancement of modulational instabilities due to combination of partial coherence and quantum corrections [20]. We refer to [12, 16, 20] for more background in physics.

For the case $\varepsilon = \mathcal{O}(1)$, in recent years, there are many results on the QZS (1.1) from the physical and mathematical points of view [8, 10, 15–18, 22–24, 30]. Particularly, Haas and Shukla [17] pointed out that the quantum corrections induced qualitative and quantitative changes, inhibiting singularities and allowing for oscillations of the width of the Langmuir envelope field. Misra *et al.* [22] revealed that the system is destabilized via a supercritical hopf-bifurcation, and periodic, chaotic, and hyperchaotic behaviors of the Fourier-mode amplitudes are identified by the analysis of Lyapunov exponent spectra and the power spectrum. For the well-posedness, the QZS is locally well-posed in L^2 data for dimension up to eight and globally well-posed for $1 \le d \le 5$ [8], which is different from the classical ZS, while the local and global well-posedness of the corresponding Cauchy problem is known only for $1 \le d \le 3$ [11, 13]. Moreover, the QZS is globally well-posed for initial data $(E_0, N_0^\varepsilon, N_1^\varepsilon) \in H^k \times H^{k-1} \times H^{k-3}$ with $k \ge 2$ and $d = 1, 2, 3$ without any size constraints on the initial data [15]. This suggests that including some more physical effects in the equations which results as a more complicated system may make the mathematical understanding much easier.

For the numerical part, there are few results for the QZS (1.1). Xiao *et al.* [29] developed a conservative finite difference scheme for the modified ZS with high-order space fractional quantum correction. Recently, Baumstark and Schratz [3] presented a new class of asymptotic preserving trigonometric integrators for QZS, and the scheme converges to the classical ZS in the limit $\lambda \to 0$ uniformly in the time discretization parameter. In [32], we proposed and analysed a highly accurate conservative method for solving the QZS, which is fourth-order accurate in space and second-order accurate in time. Zhang [33] developed a fully explicit and efficient method by applying a time splitting technique and an exponential integrator for time integration combined with the Fourier pseudospectral method in space for the QZS. Some interesting dynamical phenomena was also included.

For the QZS (1.1) in the subsonic limit regime, i.e., $\varepsilon \to 0^+$, Fang *et al.* [11] proved the solution of the corresponding fourth-order Schrödinger part converges to the solution of quantum modified nonlinear Schrödinger equation (QM-NLSE)

$$\begin{cases} i\partial_t E - (-\Delta + \lambda^2\Delta^2)E + I_\lambda(|E|^2)E = 0, & x \in \mathbb{R}^d, \quad t > 0, \\ E(x,0) = E_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.2}$$

and $E^\varepsilon(x,t) \to E(x,t)$, where $I_\lambda = (I - \lambda^2\Delta)^{-1}$ and I is the identity operator. Convergence rates of the subsonic limit regime from the QZS (1.1) to the QM-NLSE (1.2) and initial layers, as well as the propagation of oscillatory waves, have been rigorously investigated in the literatures [6,7,9,15]. Fang *et al.* [9] studied the existence and the stability of the standing waves of the QZS (1.1) for $1 \le d \le 3$. The low-regularity global well-posedness of the subsonic limit and its semi-classical limit ($\lambda \to 0$) were studied in [6]. Particularly the solution of the QZS exhibits highly oscillatory initial layers caused by the incompatibility of the initial data. This high oscillation