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GENERALIZED JACOBI SPECTRAL GALERKIN METHOD FOR FRACTIONAL-ORDER VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITH WEAKLY SINGULAR KERNELS*

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Abstract

For fractional Volterra integro-differential equations (FVIDEs) with weakly singular kernels, this paper proposes a generalized Jacobi spectral Galerkin method. The basis functions for the provided method are selected generalized Jacobi functions (GJFs), which can be utilized as natural basis functions of spectral methods for weakly singular FVIDEs when appropriately constructed. The developed method's spectral rate of convergence is determined using the L^∞-norm and the weighted L^2-norm. Numerical results indicate the usefulness of the proposed method.

Mathematics subject classification: 65L05, 65L20, 65L50.
Key words: Generalized Jacobi spectral Galerkin method, Fractional-order Volterra integro-differential equations, Weakly singular kernels, Convergence analysis.

1. Introduction

We consider in this paper the following fractional-order Volterra integro-differential equations with weakly singular kernels:

$$\begin{cases} {}_0D_t^\mu y(t) = y(t) + \displaystyle\int_0^t (t-\tau)^{-\nu} K(t,\tau) y(\tau) d\tau + g(t), \quad t \in I := [0,T], \\ y(0) = y_0, \end{cases} \tag{1.1}$$

where $0 < \mu < 1$, $0 < \nu < 1$, $K \in C(D)$ with $D := \{(t,\tau) : 0 \leqslant \tau \leqslant t \leqslant T\}$ and $g(t)$ is a continuous function. ${}_0D_t^\mu$ denotes the left-sided Reimann–Liouville fractional derivative of order μ (see the definition in (2.2a)).

The fractional calculus has made significant progress in both theory and practice, and its appearance and development have somewhat compensated for the shortcomings of the integer-order classical calculus. Fluid flow in porous materials, anomalous diffusion transport, acoustic

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wave propagation in viscoelastic materials, dynamics in self-similar structures, signal processing, financial theory, electric conductance of biological systems have all been better described using fractional calculus in the last two decades (see, e.g., [9,16,19,21,23]). In mathematical modeling of many physical phenomena, such as heat conduction in materials with memory in [20], many fractional-order Volterra integro-differential equations are used. Conduction, convection, and radiation are all examples of such equations (see, e.g., [1,3,22] and the references therein).

The three main difficulties in solving FVIDEs with weakly singular kernels in (1.1) are:

(i) fractional derivatives and integral operators are non-local operators, resulting in full matrices;

(ii) their solutions are often singular, making polynomial based approximations inefficient;

(iii) the solutions are usually singular near $t = 0$.

Spectral methods have been frequently utilized for numerical approximations [2,11,27,28] because it can provide extremely precise numerical approximations with fewer degrees of freedom. Well-designed spectral methods, in particular, appear to be particularly appealing for dealing with the above-mentioned challenges connected with FVIDEs with weakly singular kernels. For FVIDEs, polynomial based spectral methods have been developed (cf. [12,25,32,33,38] and the references therein). These approaches, on the other hand, rely on polynomial basis functions, which are not ideal for FVIDEs with non-smooth solutions at $t = 0$. Jacobi poly-fractonomials, which are defined as eigenfunctions of a fractional Sturm-Liouville problem, were first presented by Zayernouri and Karniadakis [35] to approximate the singular solutions. Chen *et al.* [4] employed generalized Jacobi functions (GJFs), which contain Jacobi poly-fractonomials as special cases, to create efficient Petrov-Galerkin methods for fractional PDEs. Following that, other authors devised spectral methods for fractional PDEs utilizing nodal GJFs [13,15,34,36,37].

When $\mu = 1$, the Eq. (1.1) is the classical Volterra integro-differential equations (VIDEs) with weakly singular kernels. Recently, many kinds of spectral collocation methods are proposed for solving Volterra integro equations (VIEs) with smooth kernels (cf. [7,17,29–31] and the references therein). To solve VIEs with weakly singular kernels, many attempts have been made to overcome the difficulties caused by the singularities of the solutions. For weakly singular VIEs, Chen and Tang [5,6] established spectral collocation methods. In [26], nonlinear VIEs with weakly singular kernels are solved using a generalized Jacobi-Galerkin method. In [18], linear VIDEs have been solved by Petrov–Galerkin method. Huang *et al.* [14] studied the supergeometric convergence of spectral collocation methods for weakly singular Volterra/Fredholm integral equations, etc.

The organization of this paper is as follows. In the next section, we introduce some useful properties of fractional calculus. In Section 3, we present the generalized Jacobi spectral Galerkin method for FVIDEs with weakly singular kernels in (1.1). Some useful lemmas for the convergence analysis are provided in Section 4. The convergence of the generalized Jacobi spectral Galerkin method is given in Section 5. We present in Section 6 some illustrative numerical results. Some concluding remarks are given in the last section.

2. Preliminaries

2.1. Fractional derivatives

We start with some preliminary definitions of fractional derivatives (see, e.g., [9,23]). To fix the idea, we restrict our attentions to the interval $\Lambda := [-1, 1]$.