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MODIFIED STOCHASTIC EXTRAGRADIENT METHODS FOR STOCHASTIC VARIATIONAL INEQUALITY*

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Abstract

In this paper, we consider two kinds of extragradient methods to solve the pseudo-monotone stochastic variational inequality problem. First, we present the modified stochastic extragradient method with constant step-size (MSEGMC) and prove the convergence of it. With the strong pseudo-monotone operator and the exponentially growing sample sequences, we establish the R-linear convergence rate in terms of the mean natural residual and the oracle complexity $O(1/\epsilon)$. Second, we propose a modified stochastic extragradient method with adaptive step-size (MSEGMA). In addition, the step-size of MSEGMA does not depend on the Lipschitz constant and without any line-search procedure. Finally, we use some numerical experiments to verify the effectiveness of the two algorithms.

Mathematics subject classification: 47J20, 90C33, 90C25.
Key words: Stochastic variational inequality, Pseudo-monotone, Modified stochastic extragradient methods, Adaptive step-size.

1. Introduction

There is a broad range of source problems from engineering, economics and finance that can be modeled as a variational inequality [1,2], which is denoted as VI(X,F) or simply VI: finding a vector $x^* \in X$ such that

$$\langle F\left(x^*\right), x - x^*\rangle \geq 0, \quad \forall x \in X,$$

where $F : X \to \mathbb{R}^n$ is a mapping and $X \subset \mathbb{R}^n$ is a nonempty closed convex set. VI involves many problems, including optimization problems, Nash games over continuous strategy sets, economic equilibrium problems, complementarity problems and so on [1,3,4]. Many effective methods for solving VI have been proposed in the past. The projection method is one of the simplest method, namely

$$x^{k+1} = \Pi_X\left[x^k - \alpha F(x^k)\right].$$

Under the assumptions on the strong monotonicity, Lipschitz continuity of F, and suitable choice of step-size α, the method generates a sequence converging to a solution [1,4–6]. To relax

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the strong monotonic hypothesis of the method, Korpelevich [7] and Antipin [8] introduced the Extra-Gradient Method (EGM). The method requires two projections per iteration

$$\begin{cases} z^k = \Pi_X \left[x^k - \alpha F(x^k)\right], \\ x^{k+1} = \Pi_X \left[x^k - \alpha F(z^k)\right]. \end{cases}$$

It converges to a solution when F is Lipschitz continuous and pseudo monotone with $\alpha \in (0, 1/L)$. Note that it is difficult to choose an appropriate step-size α when the Lipschitz constant L is unknown. Later some methods based on adaptive step-size were proposed [9–11] to overcome the disadvantage. The EGM requires to compute two projections onto the feasible set X and two values of operator F over each iteration. It is time-consuming, especially when X has a complex structure or the scale of the problem is large. Some scholars considered how to simplify the second projection of EGM [11–15]. Popov [16] proposed a modification of the EGM, where the operator is evaluated only once per iteration. Recently Hieu *et al.* [5] proposed a modified EGM, where the step-size only needs a simple computation. Hieu *et al.* [17] extended this kind of methods to infinite dimensional Hilbert spaces. In [5], the operator is evaluated only once at each iteration, and the second projection is onto the half space.

Yet lots of research on VI is restricted to deterministic regimes. However, the data of many problems were obtained under uncertain conditions in real world. Stochasticity should be taken into account for their VI formulations [3, 6, 18–20]. The stochastic variational inequality is a natural extension of deterministic VI. For a nonempty closed and convex set $X \subset \mathbb{R}^n$, $T : X \to \mathbb{R}^n$ is given by $T(x) = \mathbb{E}\left[F(x, \xi)\right]$ for any $x \in X$, the stochastic variational inequality, denoted as SVI(X,T), or simply SVI, requires to find a vector $x^* \in X$ such that

$$\langle T(x^*), x - x^* \rangle \geq 0, \quad \forall x \in X.$$

An important methodology for SVI is Stochastic Approximation (SA) approach [21, 22], where samples are obtained in an online fashion. The seminal work of SA is from Robbins and Monro [23]. Jiang and Xu [18] first proved the almost sure convergence under either strong/strict monotonicity or a variant of the acute angle condition. In addition, Yousefian *et al.* [24] studied SVI with strongly monotone but not necessarily Lipschitz continuous mappings and proved the convergence in an almost sure sense. It also displayed the rate of convergence in mean squared error. Koshal *et al.* [25] proposed two classes of SA methods: stochastic iterative Tikhonov regularization method and stochastic iterative proximal-point method. Recently, the EGM is extended to solve the SVI problems. Yousefian *et al.* [26] developed extragradient-based robust smoothing schemes for monotone SVI in non-Lipschitz continuous regimes. Kannan *et al.* [27] proposed the optimal stochastic extragradient schemes for pseudo-monotone SVI problems and their variants. The authors used the diminishing step-size and fixed samples. Iusem *et al.* [28] proposed a dynamic sampled stochastic approximated EGM for pseudo-monotone and Lipschitz continuous SVIs as the following forms:

$$\begin{cases} z^k = \Pi_X \left[x^k - \dfrac{\alpha_k}{N_k} \displaystyle\sum_{j=1}^{N_k} F\big(x^k, \xi_j^k\big)\right], \\ x^{k+1} = \Pi_X \left[x^k - \dfrac{\alpha_k}{N_k} \displaystyle\sum_{j=1}^{N_k} F\big(z^k, \eta_j^k\big)\right]. \end{cases} \tag{1.1}$$