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AN EFFICIENT FINITE DIFFERENCE METHOD FOR STOCHASTIC LINEAR SECOND-ORDER BOUNDARY-VALUE PROBLEMS DRIVEN BY ADDITIVE WHITE NOISES*

Mahboub Baccouch
Department of Mathematics, University of Nebraska at Omaha, Omaha NE 68182, USA
Email: mbaccouch@unomaha.edu

Abstract

In this paper, we develop and analyze a finite difference method for linear second-order stochastic boundary-value problems (SBVPs) driven by additive white noises. First we regularize the noise by the Wong-Zakai approximation and introduce a sequence of linear second-order SBVPs. We prove that the solution of the SBVP with regularized noise converges to the solution of the original SBVP with convergence order $\mathcal{O}(h)$ in the mean-square sense. To obtain a numerical solution, we apply the finite difference method to the stochastic BVP whose noise is piecewise constant approximation of the original noise. The approximate SBVP with regularized noise is shown to have better regularity than the original problem, which facilitates the convergence proof for the proposed scheme. Convergence analysis is presented based on the standard finite difference method for deterministic problems. More specifically, we prove that the finite difference solution converges at $\mathcal{O}(h)$ in the mean-square sense, when the second-order accurate three-point formulas to approximate the first and second derivatives are used. Finally, we present several numerical examples to validate the efficiency and accuracy of the proposed scheme.

Mathematics subject classification: 65C30, 65L12, 65L20, 60H35, 39A50.
Key words: Boundary-value problems, Finite-difference method, Additive white noise, Wiener process, Mean-square convergence, Wong-Zakai approximation.

1. Introduction and Problem Statement

Stochastic differential equations (SDEs) are differential equations where one or several terms are stochastic processes. They are used in finance (such as interest rate and stock prices), biology (such as population and epidemics), physics (such as particles in fluids and thermal noise), and control and signal processing (such as controller and filtering). Many SDE models involve the derivative of Brownian motion, also known as white noise. In this paper, we propose and analyze a stochastic finite difference (SFD) method for scalar stochastic linear second-order boundary-value problems (BVPs) driven by additive white noises

$$u'' = p(x)u' + q(x)u + r(x) + g(x)\dot{W}(x), \quad x \in [a,b], \quad u(a) = \alpha, \quad u(b) = \beta, \tag{1.1}$$

where $p(x), q(x), r(x)$, and $g(x)$ are sufficiently smooth functions on $[a,b]$. Here, α and β are real values and $\dot{W}$ is the standard one-parameter family white noise. The white noise is a generalized function or a distribution and it can be written informally as $\dot{W}(x) = dW(x)/dx$ in the sense of distribution, where $W(x)$ is the one-dimensional real-valued standard Brownian motion (or

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Wiener process) defined on some complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\{\mathcal{F}_x\}_{a\leq x\leq b}$ satisfying the usual conditions (the filtration is right-continuous and contains all P-null sets in $\mathcal{F}$) and carrying a standard one-dimensional Brownian motion W.

In our analysis, we impose the restriction as in the following deterministic case:

$$0 < \sigma = \min_{x\in[a,b]} q(x) \leq q(x), \quad x \in [a, b]. \tag{1.2}$$

A solution to (1.1) is defined as follows: Let $v = u'$. Then (1.1) can be written as

$$\begin{aligned} &u' = v, \\ &v' = p(x)v + q(x)u + r(x) + g(x)\dot{W}(x), \quad x \in [a, b], \quad u(a) = \alpha, \quad u(b) = \beta. \end{aligned}$$

This system is a formal abbreviation of the following integral equations:

$$u(x) = u(a) + \int_a^x v(y)dy, \tag{1.3a}$$

$$v(x) = v(a) + \int_a^x \big(p(y)v(y) + q(y)u(y) + r(y)\big)dy + \int_a^x g(y)dW(y) \tag{1.3b}$$

for $x \in [a, b]$ with the boundary conditions $u(a) = \alpha$ and $u(b) = \beta$. We note that second integral in (1.3b) is an Itô stochastic integral with respect to W. Unlike the integral in (1.3a) and the first integral in (1.3b), it cannot be defined as a Riemann-Stieltjes integral since the Brownian paths are of unbounded variation on any finite interval $[a, x]$ for $x > a$.

Deterministic differential equations are important tools for describing many real-world problems. Nowadays, SDEs are widely used in modeling many dynamical systems influenced by random noises. Examples of random exterior influences include trajectories, measurements, noisy signals, etc. SDEs are very powerful tools to describe real problems with uncertainties. Uncertainties may be originated from various sources such as initial/boundary conditions, geometry representation of the domain, and input parameters. SDEs arise in several areas including finance, chemistry, economics, population dynamics and genetics, physics, biology, fluid flows in random media, neuroscience, and many others. We refer the reader to [18, 26, 30–33, 37] for further applications as well as theoretical and numerical methods for SDEs.

The analytic solution of (1.1) for general coefficients p, q, r, and g is not available, and one has to use a numerical method to approximate it. A huge progress has been made over the last decades in numerical methods for stochastic initial-value problems (SIVPs). We refer the reader to [1, 2, 8–10, 24–29, 34–36, 40] and the references therein, just to mention a few. In contrast, numerical methods for stochastic two-point BVPs and stochastic partial differential equations (SPDEs) have received much less attention [2–6, 12–17, 20, 41–43]. Nevertheless, there are some numerical methods proposed in the literature for approximating the solution to (1.1). In particular, Arciniega and Allen [3–5] designed a stochastic shooting method to approximate solutions to SBVPs. The stochastic shooting method is similar to standard shooting method for deterministic BVPs. It consists of converting the SBVP to a family of SIVPs, which can be approximated with the standard numerical methods for SIVPs (such as the Euler-Maruyama method or Milstein's method). The stochastic shooting methods are easy to implement, however they require solutions to a family of SIVPs and nonlinear equations. Allen *et al.* [2] applied the finite difference and finite element methods to approximate solutions to linear parabolic and elliptic SPDEs driven by additive white noise. They first approximated the white noise process $\dot{W}$ with a piecewise constant random process. The solution of the SBVP with regularized noise is shown to converge to the solution of the original problem in the mean-square sense.