Journal of Computational Mathematics
Vol.42, No.2, 2024, 454–499.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2209-m2021-0038

CONSERVATIVE CONFORMING AND NONCONFORMING VEMS FOR FOURTH ORDER NONLINEAR SCHRÖDINGER EQUATIONS WITH TRAPPED TERM*

Meng Li and Jikun Zhao[1)]
School of Mathematics and Statistics, Zhengzhou University, Zhengzhou 450001, China
Email: limeng@zzu.edu.cn, jkzhao@zzu.edu.cn
Zhongchi Wang
School of Computer Science and Engineering, Beihang university, Beijing 100083, China
Email: wangzc_1997@outlook.com
Shaochun Chen
School of Mathematics and Statistics, Zhengzhou University, Zhengzhou 450001, China
Email: shchchen@zzu.edu.cn

Abstract

This paper aims to construct and analyze the conforming and nonconforming virtual element methods for a class of fourth order nonlinear Schrödinger equations with trapped term. We mainly consider three types of virtual elements, including H^2 conforming virtual element, C^0 nonconforming virtual element and Morley-type nonconforming virtual element. The fully discrete schemes are constructed by virtue of virtual element methods in space and modified Crank-Nicolson method in time. We prove the mass and energy conservation, the boundedness and the unique solvability of the fully discrete schemes. After introducing a new type of the Ritz projection, the optimal and unconditional error estimates for the fully discrete schemes are presented and proved. Finally, two numerical examples are investigated to confirm our theoretical analysis.

Mathematics subject classification: 65N35, 65N12, 76D07, 65N15.
Key words: H^2 conforming virtual element, C^0 nonconforming virtual element, Morley-type nonconforming virtual element, Nonlinear Schrödinger equation, Conservation, Convergence.

1. Introduction

Nonlinear Schrödinger equations (NLS) have been used to describe many physical phenomena, such as fluid dynamics, nonlinear optics, quantum physics, plasma physimas and Bose-Einstein condensates. The NLS equations have many different formulations, which have been studied by many researchers, including the analytical aspects [1–8] and the numerical aspects [9–25]. Among them, Huo and Jia [8] studied the Cauchy problem of fourth-order NLS equation related to the vortex filament. Hong and Kong [19] considered the multi-symplectic Runge-Kutta methods and multi-symplectic Fourier spectral methods for the fourth-order NLS equations with trapped term. In [20], Kong *et al.* discussed the symplectic integrator for the numerical solution of the high-order Schrödinger equation with trapped terms. The authors

* Received February 3, 2021 / Revised version received April 8, 2021 / Accepted September 19, 2022 / Published online February 23, 2023 /
1) Corresponding author

in [22, 23] considered two kinds of finite difference schemes for a class of NLS equations of high order.

This work mainly considers a class of fourth order NLS equations with trapped term

$$iu_t + \Delta^2 u + \gamma(\mathbf{x})\frac{\partial \hbar(|u|^2)}{\partial |u|^2}u + \zeta u = 0, \qquad \mathbf{x} \in \Omega, \quad (\mathbf{x}, t) \in \Omega \times (0, T], \tag{1.1}$$

$$u(\mathbf{x}, t) = \frac{\partial u}{\partial \mathbf{n}} = 0, \qquad (\mathbf{x}, t) \in \partial\Omega \times (0, T], \tag{1.2}$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{1.3}$$

where $i^2 = -1$, $\hbar(|u|^2)$ is a bounded real differentiable functional of $|u|^2 \in \mathbb{R}$, $\gamma(\mathbf{x})$ and ζ are given real-valued bounded functions, $\Omega \subseteq \mathbb{R}^2$ is a bounded domain with the boundary $\partial\Omega$, and $u_0(\mathbf{x})$ is a given complex-valued function. The trapped term ζ has two options: $\zeta(\mathbf{x})$ and $\zeta(\mathbf{x}, t)$, whose effects are to position the solutions near the boundary domain. The model (1.1)-(1.3) has important effects, such as nonlinear optics, dispersion, plasma physics, superconductivity, quantum mechanics and parabolic potential of trapping type. The model (1.1)-(1.3) keeps the charge conservation law

$$Q(t) = \int_\Omega |u(\mathbf{x}, t)|^2 d\mathbf{x} = \int_\Omega |u_0(\mathbf{x})|^2 d\mathbf{x} = Q(0). \tag{1.4}$$

If $\zeta = \zeta(\mathbf{x})$, the model (1.1)-(1.3) conserves the energy

$$\begin{aligned} E(t) &= \int_\Omega \Big[|\Delta u|^2 + \gamma(\mathbf{x})\hbar(|u|^2) + \zeta(\mathbf{x})|u|^2\Big] d\mathbf{x} \\ &= \int_\Omega \Big[|\Delta u_0|^2 + \gamma(\mathbf{x})\hbar(|u_0|^2) + \zeta(\mathbf{x})|u_0|^2\Big] d\mathbf{x} = E(0). \end{aligned} \tag{1.5}$$

As we all know, the conservative schemes always perform better than the nonconservative ones. They can preserve some invariant properties, and thus capture some physical procedures with more details. From this point of view, the discrete schemes keeping some invariant properties of the original continuous model can judge the success of the numerical simulation.

In this work, we intend to study the conservative virtual element methods (VEMs), including conforming and nonconforming ones, for the model (1.1)-(1.3). VEM can be regarded as a generalization of the classical finite element method to general meshes consisting of polygonal meshes. The different kinds of VEMs have aroused significant attentions as the result of the high flexibility of mesh handing technique and the constructed scheme without any explicit construction of the discrete shape functions. The authors in [26, 27] introduced the basic principle of VEMs for the elliptic problem and the plate bending problem. Different from the classical finite element method, the virtual element (VE) space consists of polynomial and non-polynomial functions, and its corresponding degrees of freedom must be carefully chosen so that the stiffness matrix can be calculated without actually calculating the non-polynomial functions. VEM only requires the knowledge of a polynomial subspace of the local discrete functional space to supply stable and accurate numerical methods. Hence, we can handle meshes with general shaped elements avoiding the explicit evaluation of the shape functions. In recent years, the analytical techniques of VEMs have been developed widely. For instance, a modified version for the conforming VEM was introduced in [28] which allows the exact computations of the L^2 projections on all polynomials of degree l. The nonconforming VEM was proposed in [29], for the numerical approximation of velocity and pressure in the steady Stokes problem. Then, the enhanced