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UNCONDITIONAL ERROR ANALYSIS OF VEMS FOR A GENERALIZED NONLINEAR SCHRÖDINGER EQUATION*

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Abstract

In this work, we focus on the conforming and nonconforming leap-frog virtual element methods for the generalized nonlinear Schrödinger equation, and establish their unconditional stability and optimal error estimates. By constructing a time-discrete system, the error between the solutions of the continuous model and the numerical scheme is separated into the temporal error and the spatial error, which makes the spatial error τ-independent. The inverse inequalities in the existing conforming and new constructed nonconforming virtual element spaces are utilized to derive the L^∞-norm uniform boundedness of numerical solutions without any restrictions on time-space step ratio, and then unconditionally optimal error estimates of the numerical schemes are obtained naturally. What needs to be emphasized is that if we use the pre-existing nonconforming virtual elements, there is no way to derive the L^∞-norm uniform boundedness of the functions in the nonconforming virtual element spaces so as to be hard to get the corresponding inverse inequalities. Finally, several numerical examples are reported to confirm our theoretical results.

Mathematics subject classification: 65N35, 65N12, 76D07, 65N15.
Key words: Conforming and nonconforming, Virtual element methods, Leap-frog scheme, Generalized nonlinear Schrödinger system, Unconditionally optimal error estimates.

1. Introduction

Consider a generalized nonlinear Schrödinger (GNLS) system

$$iu_t + \Delta u + \psi f(|u|)u + l(|u|)u = 0, \quad (\mathbf{x},t) \in \Omega \times (0,T], \tag{1.1}$$

$$\kappa_1 \psi - \kappa_2^2 \Delta \psi = f(|u|)|u|^2, \quad (\mathbf{x},t) \in \Omega \times (0,T], \tag{1.2}$$

$$u(\mathbf{x},t) = 0, \quad \psi(\mathbf{x},t) = 0, \quad (\mathbf{x},t) \in \partial\Omega \times (0,T], \tag{1.3}$$

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \overline{\Omega}, \tag{1.4}$$

where $i^2 = -1$, $\kappa_1 \geq 0$ and κ_2 are real constants satisfying $\kappa_1 + \kappa_2 \neq 0$, and u_0, f and l are two given real-valued continuous functions.

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The system (1.1)-(1.4) under different parameters indeed reduces to many classical models. For instance, the system is indeed a class of nonlinear Schrödinger equations with different nonlinear terms when $\kappa_2 = 0$ [4, 49, 51]. If $\kappa_1 = 0$, $l(|u|) = 0$ and $\kappa_2 \neq 0$, the system reduces to the Schrödinger-Poisson model [36, 47, 48]. When $\kappa_1 = 0$, $\kappa_2 \neq 0$ and $f(|u|)$ is a constant, the system is called the Schrödinger-Poisson-Slater model [28, 65]. In addition, if $l(|u|) = 0$, the model (1.1)-(1.4) is degraded to the Schödinger-Helmholtz system [24, 33]. The system (1.1)-(1.4) could be used to describe many physical phenomena in the field of quantum optics, quantum mechanics, and plasma physics, which motivates scholars to study the model along the mathematical and numerical points. On the mathematical level, Leo and Rial [36] studied the well-posedness of the Cauchy problem for the Schrödinger-Poisson equation. Bao *et al.* [10] obtained the formal derivation, existence and uniqueness analysis of the Schrödinger-Poisson-Xα model with/without an external potential. Trabelsi [57] studied the existence and uniqueness global in time of solutions of the coupled higher-order Schrödinger-Poisson-Slater equations with a self-consistent Coulomb potential. In [35], the existence and stability of standing waves for the Schrödinger-Poisson-Slater equation were considered. More detailed mathematical analysis for different Schrödinger-type models can be found in [16, 24, 48, 55]. On the numerical level, Liao *et al.* [45] studied a fourth-order compact difference scheme two-dimensional linear Schrödinger equations with periodic boundary conditions. Zhang [64] considered the compact finite difference methods for the Schrödinger-Poisson equation in a bounded domain and established their optimal error estimates under proper regularity assumptions on wave function and external potential. Akrivis *et al.* [3] studied the fully discrete Galerkin finite element methods (FEMs) of second-order temporal accuracy for the nonlinear Schrödinger equation. In [33], a Crank-Nicolson FEM for a class of nonlinear Schrödinger-Helmholtz system was constructed, and unconditional stability and convergence of the proposed scheme were studied by using time-space error splitting technique. Some other numerical methods have been applied for the GNLS system, such as finite difference methods [45,59], spectral methods [10,27], discontinuous Galerkin method [34, 46], virtual element methods (VEMs) [44], and so on.

VEM, regarded as an extension of FEM, different from polygonal FEM [54], composite FEM [31] and extended/generalized FEM [29], possesses great flexibility in utilizing meshes with almost arbitrary polygons and polyhedrons [11]. As the basic principle of the VEM introduced in the pioneering works [11,20], virtual element space comprises polynomial and non-polynomial functions, and the numerical schemes dispense with any explicit construction of the discrete shape functions. In order to calculate the corresponding stiffness matrix without actually calculating the non-polynomial function, degrees of freedom should be selected attentively. VEM only requires the information of a polynomial subspace of the local discrete function space to offer stability and accuracy, as the result that more general shaped elements can be used and explicit evaluation of the shape functions is avoided. Recently, various VEMs have been widely used in various physical models, such as elliptic problem [7, 21], elasticity problem [12, 63], Stokes/Navier-Stokes models [5, 14, 15, 66], Cahn-Hilliard equation [6], $2m$-th order partial differential equations [26], reaction-subdiffusion equations [42, 43], and so on. However, we have not found the VEM and FEM for the system (1.1)-(1.4) that is the initial research motivation of this work. We emphasize that FEM and VEM are good choices for the system (1.1)-(1.4) because they allow us to work in very low regularity regimes that cannot be handled with FDMs or spectral methods.

Most recently, we studied the conforming and nonconforming VEMs for the two dimensional nonlinear Schrödinger equations [44]. This work tried to derive the optimal rate of convergence