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UNCONDITIONAL STABILITY OF A CRANK-NICOLSON ADAMS-BASHFORTH 2 NUMERICAL METHOD

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Abstract. Nonlinear partial differential equations modeling turbulent fluid flow and similar processes present special challanges in numerical analysis. Regions of stability of implicit-explicit methods are reviewed, and an energy norm based on Dahlquist's concept of G-stability is developed. Using this norm, a time-stepping Crank-Nicolson Adams-Bashforth 2 implicit-explicit method for solving spatially-discretized convection-diffusion equations of this type is analyzed and shown to be unconditionally stable.

Key words. convection-diffusion equations, unconditional stability, IMEX methods, Crank-Nicolson, Adams-Bashforth 2

1. Introduction

The motivation of this work is to consider the stability of numerical methods when applied to ordinary differential equations (ODEs) of the form

$$u'(t) + Au(t) - Cu(t) + B(u)u(t) = f(t), \tag{1}$$

in which $A, B(u)$ and C are $d \times d$ matrices, $u(t)$ and $f(t)$ are d-vectors, and

$$A = A^T \succ 0, B(u) = -B(u)^T, C = C^T \succcurlyeq 0 \text{ and } \quad A - C \succ 0. \tag{2}$$

Here $\succ$ and $\succcurlyeq$ denote the positive definite and positive semidefinite ordering, respectively.

Models of the behavior of turbulent fluid flow using convection-diffusion partial differential equations discretized in the spatial variable give rise to a system of ODEs, such as

$$\dot{u}_{ij}(t) + b \cdot \nabla^h u_{ij} - (\epsilon_0(h) + \nu)\Delta^h u_{ij} + \epsilon_0(h)\Delta^h u_{ij} = f_{ij}, \tag{3}$$

where Δ^h is the discrete Laplacian, ∇^h is the discrete gradient, and $\epsilon(h)$ is the artificial viscosity parameter. System (3) is of the form (1) and (2), where

$$A = -(\epsilon_0(h) + \nu)\Delta^h, \quad C = -\epsilon_0(h)\Delta^h, \quad B(u) = b \cdot \nabla^h.$$

In this case the matrix $B(\cdot)$ is constant, but in general it may depend on u, and thus the system is allowed to have a nonlinear part. A linear multistep method for the numerical integration a system $u'(t) = F(t, u)$, such as (1), is

$$\sum_{j=-1}^{k} \alpha_j u_{n-j} = \Delta t \sum_{j=-1}^{k} \beta_j F_{n-j}, \tag{4}$$

where t is defined on $\mathcal{I} = [t_0, t_0 + T] \subset \mathbb{R}$, $u_{n-j} \in \mathbb{R}^d$, $F_{n-j} = F(t_{n-j}, u_{n-j})$.

This work will discuss the regions of stability for implicit-explicit (IMEX) methods applied to systems of the form (1), and prove that unconditional stability (the

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method's stability properties are independent of the choice of step-size Δt) holds for a proposed Crank-Nicolson Adams-Bashforth 2 (CNAB2) IMEX numerical method,

$$\begin{aligned}
&\frac{u_{n+1}-u_n}{\Delta t} + (A-C)^{\frac{1}{2}} \\
(5)\quad &\times \Big(A(A-C)^{-\frac{1}{2}}\tfrac{1}{2}u_{n+1} + \big(\tfrac{1}{2}A - \tfrac{3}{2}C\big)(A-C)^{-\frac{1}{2}}u_n + \tfrac{1}{2}C(A-C)^{-\frac{1}{2}}u_{n-1}\big)\Big) \\
&+ B(\mathcal{E}_{n+\frac{1}{2}})(A-C)^{-\frac{1}{2}}\Big(\tfrac{1}{2}A(A-C)^{-\frac{1}{2}}u_{n+1} + \big(\tfrac{1}{2}A - \tfrac{3}{2}C\big)(A-C)^{-\frac{1}{2}}u_n \\
&+ \tfrac{1}{2}C(A-C)^{-\frac{1}{2}}u_{n-1}\big)\Big) = f_{n+\frac{1}{2}},
\end{aligned}$$

where $\mathcal{E}_{n+\frac{1}{2}} = \frac{3}{2}u_n + \frac{1}{2}u_{n-1}$, an explicit approximation of $u(t_{n+\frac{1}{2}})$. This method is a second-order convergent numerical scheme of the form (4). Section 2 discusses earlier related results for IMEX methods, and Section 3 motivates the unconditional stability analysis of (5) by deriving illustrative stability results for related scalar IMEX methods. With these results in mind, unconditional stability of method (5) is proven in Section 4. Section 5 demonstrates the theory with several numerical tests, the last of which shows the method's effectiveness when applied to a system that is a close variation of (3).

2. Previous IMEX Stability Results

In [5], Frank *et al.* consider applying IMEX methods to a system of ODEs of the form

$$u'(t) = F(t, u(t)) + G(t, u(t)),$$

where F is the stiff, and G is the non-stiff parts of the system. Considering the scalar test equation

$$u'(t) = \lambda u(t) + \gamma u(t),$$

they find that under these conditions, $\lambda \Delta t$ and $\gamma \Delta t$ lying in the regions of stability of their respective methods are sufficient conditions for the IMEX method to be assymptotically stable. As is demonstrated in Section 3, these are not necessary conditions when the system is under assymptions (2), which is due to the additional requirement that $A - C$ be positive-definite.

Akrivis *et al.* study a system of the same form as (1) except that B is assumed to be self-adjoint instead of skew-symmetric. They analyze a general class of methods that are implicit in all linear terms, and explicit in all nonlinear terms, and show these methods to be absolutely stable [1].

Finally, Anitescu *et al.* [2] show that the first-order IMEX method

$$\frac{u_{n+1}-u_n}{\Delta t} + Au_{n+1} - Cu_n + B(u)u_{n+1} = f_{n+1} \tag{6}$$

is unconditionally stable. Unlike in [1] there are two linear terms, one of which will be approximated explicitly, while the solution vector in the nonlinear term $B(u)u(t)$ is computed using an implicit scheme.

3. Stability for Scalar IMEX Methods

Consider the Cauchy problem

$$y'(t) = (\epsilon + \nu)\lambda y(t) - \epsilon \lambda y(t), \tag{7}$$

$$y : \mathbb{R} \to \mathbb{R}, \quad y(0) = 1, \quad \lambda < 0, \quad 0 < \nu, \quad 0 < \epsilon. \tag{8}$$