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Multiple Solutions for an Elliptic Equation with Hardy Potential and Critical Nonlinearity on Compact Riemannian Manifolds

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Abstract. We prove the existence of multiple solutions of an elliptic equation with critical Sobolev growth and critical Hardy potential on compact Riemannian manifolds.

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1 Introduction

Let (M,g) be a compact Riemannian manifold of dimension $n\geq 3$. For a fixed point p in M, we define a function ρ_p on M as follows

$$\rho_p(x)=\begin{cases}\mathrm{dist}_g(p,x), & x\in B(p,\delta_g),\\ \delta_g, & x\in M\setminus B(p,\delta_g),\end{cases} \tag{1.1}$$

where δ_g denotes the injectivity radius of M.

Let h and f be two regular functions on M. Consider on $M\setminus\{p\}$ the following Hardy-Sobolev equation:

$$\Delta_g u-\frac{h(x)}{\rho_p^2(x)}u=f(x)|u|^{2^*-2}u, \tag{$E_{f,h}$}$$

where $\Delta_g u=-\mathrm{div}(\nabla_g u)$ is the Laplace-Beltrami operator and $2^*=\frac{2n}{n-2}$ is the Sobolev critical exponent.

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As one may notice, when dropping the singular term $\frac{1}{\rho_p^2(x)}$ from equation $(E_{f,h})$ and putting $h=\frac{n-2}{4(n-1)}Scal_g$, where $Scal_g$ is the scalar curvature of (M,g), one falls in the celebrated prescribed scalar curvature equation whose origin comes from the study of conformal deformation of the metric to prescribed scalar curvature. A smooth positive solution u of the prescribed scalar curvature equation provides a conformal metric $g'=u^{\frac{4}{n-2}}g$ with scalar curvature the function f; when f is constant we fall n the famous Yamabe equation. The prescribed scalar curvature equation is largely studied and lot of results have been obtained. For those interested, good comprehensive references may be the books [1] and [2]. Equation $(E_{f,h})$ can be, then, seen as a singular prescribed scalar curvature equation.

The case where the function ρ_p, in equation $(E_{f,h})$, is of power $0<\gamma<2$ and $f\equiv 1$, has been studied in [3] and is related to the study of conformal deformation to constant scalar curvature of metrics which are smooth only in some geodesic ball $B(p,\delta)$ (see [3,4]). Note that the author in [3,4] considers also equation $(E_{f,h})$, with $f\equiv 1$, and shows existence of a solution on compact manifolds.

In this paper, we are interested in proving the existence of multiple solutions of equation $(E_{f,h})$. The tool used is a classical theorem from critical point theory (see Theorem 4.2 below). Note that the main difficulty in applying this theorem lies in satisfying the compactness assumption under which the critical points exist. This difficulty is due mainly to the presence of Sobolev exponent and Hardy potential. More explicitly, presence of Sobolev exponent and Hardy potential renders non-compact the inclusions $H_1^2(M)\subset L_{2^*}M$ and $H_1^2(M)\subset L(M,\rho_p^{-2})$ (see Section 2 for definition of the notation). This leads us to analyze compactness of Palais-Smale sequences which can be done by means of of a Struwe type decomposition formulas of Palais-Smale sequences.

2 Notation, useful results and statement of the main result

In this section, we introduce some notation and results that are useful in our study.

We denote by $\mathcal{D}^{1,2}(\mathbb{R}^n),(n\geq 3)$, the Euclidean Sobolev space which is the closure space of $C_o(\mathbb{R}^n)$, the space of functions u with compact support in $\mathbb{R}^n$, with respect to the norm

$$||u||_{\mathcal{D}^{1,2}(\mathbb{R}^n)}=\sqrt{\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x}.$$

Let $K(n,2)$ denote the best constant in the sharp Euclidean Sobolev inequality

$$\left(\int_{\mathbb{R}^n}|u|^{2^*}\mathrm{d}x\right)^{\frac{1}{2^*}}\leq K(n,2)\left(\int_{\mathbb{R}^n}|\nabla u|^2\right)^{\frac{1}{2}}.$$

The explicit value of $K(n,2)$ has been obtained in [5] and [6] (see also [2, Theorem 5.3.1])

$$K(n,2)=\sqrt{\frac{4}{n(n-2)w_n^{2/n}}},$$