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Uniform RIP Bounds for Recovery of Signals with Partial Support Information by Weighted ℓ_p-Minimization

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Abstract. In this paper, we consider signal recovery in both noiseless and noisy cases via weighted ℓ_p $(0<p\leq 1)$ minimization when some partial support information on the signals is available. The uniform sufficient condition based on restricted isometry property (RIP) of order tk for any given constant $t>d$ ($d\geq 1$ is determined by the prior support information) guarantees the recovery of all k-sparse signals with partial support information. The new uniform RIP conditions extend the state-of-the-art results for weighted ℓ_p-minimization in the literature to a complete regime, which fill the gap for any given constant $t>2d$ on the RIP parameter, and include the existing optimal conditions for the ℓ_p-minimization and the weighted ℓ_1-minimization as special cases.

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Key words: Compressed sensing, weighted ℓ_p minimization, stable recovery, restricted isometry property.

1 Introduction

In compressed sensing, a central goal is to efficiently recover sparse signals $\boldsymbol{x}\in\mathbb{R}^n$ from a relatively small number of linear measurements, i.e.

$$\boldsymbol{y}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{e}, \tag{1.1}$$

where $\boldsymbol{y}\in\mathbb{R}^m$, $\boldsymbol{A}\in\mathbb{R}^{m\times n}(m\ll n)$ is a sensing matrix and $\boldsymbol{e}\in\mathbb{R}^m$ denotes a vector of measurement errors. It has been a research focus in applied mathematics, statistics, and machine

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learning, with abundant applications ranging from medical imaging to speech recognition and video coding. A series of fast algorithms have been developed to recover the signal $\boldsymbol{x}$ from a relatively small number of linear measurements (1.1). The ℓ_p-minimization with $0<p\leq 1$ is among the most well-known algorithms for the reconstruction of the signal $\boldsymbol{x}$

$$\begin{aligned} &\min_{\boldsymbol{x}\in\mathbb{R}^n} \|\boldsymbol{x}\|_p^p \\ &\text{s.t.} \quad \boldsymbol{A}\boldsymbol{x}-\boldsymbol{y}\in\mathcal{B}, \end{aligned} \tag{1.2}$$

where $\mathcal{B}$ is a set determined by the noise structure and $\|\boldsymbol{x}\|_p=(\sum_{i=1}^n|x_i|^p)^{1/p}$. For the noiseless case, $\mathcal{B}=\{\mathbf{0}\}$.

In this paper, we consider the weighted ℓ_p-minimization $(0<p\leq 1)$ [7–9, 11–15, 17, 18, 20] to recover the signal $\boldsymbol{x}$ from (1.1), when some prior information is included in the estimates of the support of $\boldsymbol{x}$ or some estimates of largest coefficients of $\boldsymbol{x}$. For instance, video and audio signals exhibit strong correlation over temporal frames, which can be used to estimate a portion of the support based on previously decoded frames. The main idea inherited in the weighted ℓ_p-minimization is to make the entries of $\boldsymbol{x}$, which are expected to be large, be penalized less in the weighted objective function by introducing a weight vector $\mathbf{w}\in[0,1]^n$. The weighted ℓ_p-minimization is formulated as follows:

$$\begin{aligned} &\min_{\boldsymbol{x}\in\mathbb{R}^n} \|\boldsymbol{x}\|_{p,\mathbf{w}}^p \\ &\text{s.t.} \quad \boldsymbol{A}\boldsymbol{x}-\boldsymbol{y}\in\mathcal{B}, \end{aligned} \tag{1.3}$$

where

$$\|\boldsymbol{x}\|_{p,\mathbf{w}}=\left(\sum_{i=1}^n \mathrm{w}_i|x_i|^p\right)^{\frac{1}{p}}.$$

In particular, the weighted ℓ_p-minimization (1.3) reduces to the well-known weighted ℓ_1-minimization used for the signal recovery when $p=1$, i.e.

$$\begin{aligned} &\min_{\boldsymbol{x}\in\mathbb{R}^n} \|\boldsymbol{x}\|_{1,\mathbf{w}} \\ &\text{s.t.} \quad \boldsymbol{A}\boldsymbol{x}-\boldsymbol{y}\in\mathcal{B}. \end{aligned} \tag{1.4}$$

Let $\widetilde{T}\subseteq[n]=\{1,2,\ldots,n\}$ be a known support estimate of $\boldsymbol{x}$. The weight vector $\mathbf{w}$ in this paper is taken by

$$\mathrm{w}_i=\begin{cases} \omega, & i\in\widetilde{T}, \\ 1, & i\in\widetilde{T}^c \end{cases} \tag{1.5}$$

for some fixed $\omega\in[0,1]$.

The signal recovery based on partially known support is introduced in [2, 15, 20]. In [2, 14, 16, 19, 20], the known support information is incorporated using weighted ℓ_1-minimization with zero weights on the known support $\widetilde{T}$, i.e. $\omega=0$ in (1.5). Friedlander *et al.* [9] extended the weighted ℓ_1-minimization to nonzero weights, i.e. $\omega\in[0,1]$