Journal of Nonlinear Modeling and Analysis http://jnma-online.com
Volume 6, Number 1, March 2024, 32–55 DOI:10.12150/jnma.2024.32

# Traveling Wave of Three-Species Stochastic Lotka-Volterra Competitive System*

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**Abstract** This paper is devoted to a three-species stochastic competitive system with multiplicative noise. The existence of stochastic traveling wave solution can be obtained by constructing sup/sub-solution and using random dynamical system theory. Furthermore, under a more restrict assumption on the coefficients and by applying Feynman-Kac formula, the upper/lower bounds of asymptotic wave speed can be achieved.

**Keywords** Stochastic competitive system, white noise, traveling wave solution, asymptotic wave speed

**MSC(2010)** 35R60, 60H15.

## 1. Introduction

In this paper, we are interested in the following stochastic three-species competition model driven by Itô type multiplicative noise

$$\begin{cases} u_t = u_{xx} + u(1 - u - a_1 v - b_1 w) + \epsilon u dW_t, \\ v_t = v_{xx} + v(1 - v - a_2 u) + \epsilon(v - 1)dW_t, \\ w_t = w_{xx} + w(1 - w - b_2 u) + \epsilon(w - 1)dW_t, \\ u(0) = u_0, v(0) = 1 - \chi_{(-\infty,0]}, w(0) = 1 - \chi_{(-\infty,0]}, \end{cases} \tag{1.1}$$

where $u = u(t, x)$, $v = v(t, x)$ and $w = w(t, x)$ denote the species densities of three competing species at location $x \in R$ and time $t > 0$ respectively. Moreover, $a_i > 0$ and $b_i > 0$ represent the interspecific competition coefficients, and the environment carrying capacity of each species is ruled to be "1". Further, $W(t)$ is the white noise. Let $\epsilon = 0$, $a_2 = b_2$ and dispersal terms be replaced by nonlocal dispersal functions. Then equation (1.1) is reduced to the model proposed by Dong, Li and Wang in [2], and they showed the existence, monotonicity and asymptotic behavior of traveling waves with bistable dynamics. Based on their work, Wang, Chen and Wu [24] used a three-species competition model to expand Lotka-Volterra model to empirical analysis, and concluded that cooperative action is better than competitive strategy. Furthermore, He and Zhang [6] studied the linear determinacy of critical wave speed of three-species competitive system with nonlocal dispersal by constructing more precise conditions and suitable upper solutions. Moreover,

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*The authors were supported by National Natural Science Foundation of China (Grant Nos. 11771449, 12031020).

Liu et al., [14] studied three-species competition-diffusion model in a general case where every species competes with each other, and they pointed out that the wave speed of the slowest species is dependent on the other two faster species.

Throughout the paper, we always assume the coefficients of three-species competitive system (1.1) as follows

- (C1) $a_1 < \frac{1}{2}$, $b_1 < \frac{1}{2}$, $a_2 \geq 2$, $b_2 \geq 2$;
- (C2) $2\max\{a_1a_2, b_1b_2\} < 2 - a_1 - b_1$;
- (C3) $2\min\{a_1a_2 + b_1b_2\} + (a_1 + b_1 - 1)^2 \geq 1$;
- (C4) $\max\{a_2 - 1, b_2 - 1\} \leq \frac{1}{1-a_1-b_1}$.

Obviously, $(C1) \cap (C2) \cap (C3) \cap (C4)$ is not empty. Under condition (C1), there exist five nonnegative equilibria $P_1 = (0, 1, 1)$, $P_2 = (1, 0, 0)$, $P_3 = (0, 1, 0)$, $P_4 = (0, 0, 1)$ and $P_5 = (0, 0, 0)$, where $P_2$ is the only stable equilibrium, and the traveling wave solution is a trajectory connecting $P_1$ and $P_2$. More precisely, it reflects that the species $u$ wins the competition rather than the pair $(v, w)$.

Letting $v := 1 - \tilde{v}$, $w := 1 - \tilde{w}$ and dropping the tilde, we have

$$\begin{cases} u_t = u_{xx} + u(1 - a_1 - b_1 - u + a_1v + b_1w) + \epsilon u dW_t, \\ v_t = v_{xx} + (1 - v)(a_2u - v) + \epsilon v dW_t, \\ w_t = w_{xx} + (1 - w)(b_2u - w) + \epsilon w dW_t, \\ u(0) = \chi_{(-\infty,0]}, v(0) = \chi_{(-\infty,0]}, w(0) = \chi_{(-\infty,0]}, \end{cases} \tag{1.2}$$

and it is easy to see that (1.2) is a stochastic cooperative system, and the two equilibria $P_1$ and $P_2$ turn to be

$$\tilde{P}_1 = (0, 0, 0), \quad \tilde{P}_2 = (1, 1, 1) \tag{1.3}$$

respectively.

It is worth mentioning that most existing results for stochastic traveling wave solution deal with the scaler Fisher-KPP equation. For instance, Tribe [23] studied the KPP equation with nonlinear multiplicative noise $\sqrt{u}dW_t$, and Muëller et al., [16–18] studied the KPP equation with $\sqrt{u(1-u)}dW_t$. Both of their works take the Heaviside function as the initial data, and the main contribution of Muëller is that he explicitly described the influence brought by the noise, whether it is weak or strong, and successfully estimated the wave speed with an upper bound and a lower bound. Zhao et al., [3, 20, 21] confirmed that only if the strength of noise is moderate, and when the multiplicative noise is $k(t)dW_t$, the effects of noise would present or the solution would tend to be zero or converge to the deterministic traveling wave solution. Huang and Liu [8] studied the KPP equation driven by dual noises $k_1udW_1(t)$ and $k_2(K - u)dW_2(t)$, and revealed the bifurcations of solution induced by the strength of noise. For stochastic two-species cooperative system, Wen, Huang and Li [27] used random monotone dynamical systems and the Kolmogorov tigheness criterion to obtain the existence of stochastic traveling wave solution, and then by constructing the upper and lower solution and applying Feynman-Kac formula, they obtained the estimation of the upper bound and lower bound for wave speed respectively. The novelty of this paper not only in the three-species competitive system we study, for which there is no relevant work, but also in our confirmation that the lower bound of wave speed depends on the impact of vulnerable groups on powerful groups.