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Existence and Multiplicity of Solutions for a Biharmonic Kirchhoff Equation in $\mathbb{R}^5$*

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Abstract We consider the biharmonic equation $\Delta^2 u - \left(a + b\int_{\mathbb{R}^5} |\nabla u|^2 dx\right)\Delta u + V(x)u = f(u)$, where $V(x)$ and $f(u)$ are continuous functions. By using a perturbation approach and the symmetric mountain pass theorem, the existence and multiplicity of solutions for this equation are obtained, and the power-type case $f(u) = |u|^{p-2}u$ is extended to $p \in (2, 10)$, where it was assumed $p \in (4, 10)$ in many papers.

Keywords Biharmonic equation, multiplicity of solutions, variational method

MSC(2010) 35J85, 47J30, 49J52

1. Introduction

We consider the existence and multiplicity of solutions for the following biharmonic equation

$$\begin{cases} \Delta^2 u - \left(a + b\displaystyle\int_{\mathbb{R}^5} |\nabla u|^2 dx\right)\Delta u + V(x)u = f(u), \\ u(x) = u(|x|) \in H^2(\mathbb{R}^5), \end{cases} \tag{1.1}$$

where $V \in C(\mathbb{R}^5, \mathbb{R})$, $f \in C(\mathbb{R}, \mathbb{R})$. Biharmonic equations appear in many areas, for example, some of these problems arise from different areas of applied mathematics and physics such as surface diffusion on solids, Mircro Electro-Mechanical systems, and flow in Hele-Shaw cells (see [7]). Also, this kind of equations can describe the static deflection of an elastic plate in a fluid and the study of traveling waves in suspension bridges [6, 15]. These equations have been discussed by many authors. Indeed, if we replace $f(u)$ by $f(x, u)$ and set $V(x) = 0$, and a domain $\Omega \subset \mathbb{R}^3$, problem (1.1) becomes the following biharmonic elliptic equation of Kirchhoff type

$$\begin{cases} \Delta^2 u - \left(a + b\displaystyle\int_{\mathbb{R}^5} |\nabla u|^2 dx\right)\Delta u = f(x, u) \quad in\ \Omega, \\ u = \nabla u = 0 \quad on\ \partial\Omega, \end{cases} \tag{1.2}$$

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which is related to the general form of the following stationary analogue of the equation

$$u_{tt} + \Delta^2 u - \left(a + b\int_{\mathbb{R}^5} |\nabla u|^2 dx\right)\Delta u = f(x,u), \quad x \in \Omega. \tag{1.3}$$

Equation (1.3) is used to describe some phenomena appearing in different engineering, physical, and other scientific fields, because it is regarded as a good approximation for describing nonlinear vibrations of beams or plates [2, 4]. For example, on bounded domains, Zhang and Wei [19] used the mountain pass theorem and linking theorem to obtain the existence and multiplicity of results for the following problem

$$\begin{cases} \Delta^2 u + a\Delta u = \lambda |u|^{q-2}u + f(x,u) \quad in\ \Omega, \\ u = \nabla u = 0 \quad on\ \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, a is a constant, $f \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ and $1 < q < 2$. If $\lambda = 0$, An and Liu [1] obtained the existence of solutions of (1.4). By using critical theorems, the multiple results of (1.4) were proved in [9]. Some related results can be found in [8, 10, 16] and the references therein.

As the presence of term $\int_\Omega |\nabla u|^2$, problem (1.1) is no longer a pointwise identity and therefore, this equation is viewed as an elliptic equation coupled with non-local terms. The competing effect of the non-local term brings some mathematical challenges to the analysis, and also makes the study of such problems particularly interesting. Another difficulty lies in proving the boundedness of PS-sequences, which is very important to use variational methods. In many papers, in order to get the boundedness of PS-sequences, such as in [11], the authors need to assume $p > 4$ in $(H2)$ and the famous AR-condition. While in our paper, we relax $p > 2$ and drop the AR-condition.

In order to state our main results, we give the following hypotheses.

- $(H0)$ $V(x) = V(|x|)$ for any $x \in \mathbb{R}^5$, and $\inf_{x\in\mathbb{R}^5} V(x) := V_0 > 0$;
- $(H1)$ $f \in C(\mathbb{R}, \mathbb{R})$ and $\lim_{t\to 0} \frac{f(t)}{t} = 0$;
- $(H2)$ $\limsup_{|t|\to\infty} \frac{|f(t)|}{|t|^{p-1}} < \infty$ for some $p \in (2, 10)$;
- $(H3)$ For $\alpha \in (\frac{1}{3}, \frac{2}{5})$, $t \neq 0$, $f(t)t \geq (2 + 5\alpha)F(t) > 0$, where $F(t) = \int_{\mathbb{R}^5} f(t)dt$;
- $(H4)$ $|f(t)| \leq c_1|t| + c_2|t|^{s-1}$, $s \in (2, 10)$;
- $(H5)$ $F(-t) = F(t)$, $\forall t \in \mathbb{R}$.

Note that if $b = 0$ in problem (1.1), and it transforms into the following biharmonic equation

$$\Delta^2 u - a\Delta u + V(x)u = f(u), \tag{1.5}$$

which does not depend on the nonlocal term $\int_{\mathbb{R}^5} |\nabla u|^2$ any more. In contrast to problem (1.5), the nonlocal term makes problem (1.1) more complex in finding sign-changing solutions. The main difficulties are as follows:

(1) We don't have the following decomposition

$$\hat{I}(u) = \hat{I}(u^+) + \hat{I}(u^-), \quad \langle I'(u), u^\pm\rangle = \langle I'(u^\pm), u^\pm\rangle,$$

where $\hat{I}$ is the energy functional of (1.5) given by

$$\hat{I}(u) = \frac{1}{2}\int_{\mathbb{R}^5} (|\Delta u|^2 + a|\nabla u|^2 + V(x)u^2) - \int_{\mathbb{R}^5} F(u).$$