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Maximum-Principle-Preserving, Steady-State-Preserving and Large Time-Stepping High-Order Schemes for Scalar Hyperbolic Equations with Source Terms

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Abstract. In this paper, we construct a family of temporal high-order parametric relaxation Runge–Kutta (pRRK) schemes for stiff ordinary differential equations (ODEs), and explore their application in solving hyperbolic conservation laws with source terms. The new time discretization methods are explicit, large time-stepping, delay-free and able to preserve steady state. They are combined with fifth-order weighted compact nonlinear schemes (WCNS5) spatial discretization and parametrized maximum-principle-preserving (MPP) flux limiters to solve scalar hyperbolic equations with source terms. We prove that the fully discrete schemes preserve the maximum principle strictly. Through benchmark test problems, we demonstrate that the proposed schemes have fifth-order accuracy in space, fourth-order accuracy in time and allow for large time-stepping without time delay. Both theoretical analyses and numerical experiments are presented to validate the benefits of the proposed schemes.

AMS subject classifications: 34D20, 35L65, 65M06, 65M20, 76M20

Key words: Maximum-principle-preserving, steady-state-preserving, parametric relaxation Runge–Kutta schemes, weighted compact nonlinear schemes, hyperbolic equations.

1 Introduction

Scalar hyperbolic equations with source terms arise in various applications, including fluid dynamics [1, 2], combustion [3], and chemical reactions [4–6]. Numerical methods for solving these equations are challenging because of the presence of source terms, especially when the source terms are stiff, which may lead to numerical instability and violate

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physical constraints such as the maximum principle. Traditional high-order numerical methods typically suffer from these problems, leading to excessive computational costs and numerical errors.

We consider the following scalar hyperbolic equations with source term in the multi-dimension case

$$\begin{cases} u_t+\nabla\cdot\mathbf{F}(u)=\dfrac{1}{\varepsilon}s(u), \\ u(\mathbf{x},0)=u_0(\mathbf{x}), \end{cases} \tag{1.1}$$

with $\varepsilon>0$ being a constant. We assume that the source term $s(u)$ is dissipative in sense that

$$s'(u)\leq 0, \quad s(0)=0. \tag{1.2}$$

Then the complication of solving nonlinear hyperbolic problems arises from the fact that irregularity could be developed in a short time period even when the initial data is smooth. It demands the consideration of the physically relevant weak solution, the so-called entropy solution [7]. An important property of the entropy solution to (1.1) is that it preserves the maximum principle [8].

It is widely acknowledged that the standard numerical schemes for solving (1.1) may face several challenging difficulties. One of the main difficulties is their behavior when the solution contains discontinuities. First-order schemes, for instance, are robust near discontinuities, but they are highly dissipative. High-order schemes are less dissipative and have higher resolution, but they typically generate nonphysical oscillations near discontinuities. Different approaches have been developed to eliminate these nonphysical oscillations while maintaining high-order accuracy. One such approach is total variation diminishing (TVD) schemes, which were originally developed by Harten [9]. TVD schemes prevent spurious oscillations in their solutions and maintain high accuracy in most computational domain, except at smooth extrema which they degenerate to first-order accuracy [10, 11]. The discontinuous Galerkin (DG) methods [12–15] are also highly favored as a high-order approximation technique. Another popular approach is weighted essentially non-oscillatory (WENO) schemes [16–23]. Additionally, weighted compact nonlinear schemes (WCNS) have been developed by Deng and Zhang in [24], and later improved by Deng et al. [25, 26]. Moreover, Nonomura et al. [27, 28] have obtained various high-order versions. Compared with the WENO schemes, the WCNS schemes are easier to implement because they use weighted interpolation on the original variables, and they have higher resolution at the same order of accuracy and a similar ability to capture strong shock waves [29]. Unfortunately, the same as the WENO schemes, the WCNS schemes do not naturally satisfy the maximum principle because of their high-order nonlinear Lagrange interpretation [30], which is another difficulty. To address this issue, Xu [31] proposed the parametrized MPP flux limiters for 1D scalar hyperbolic conservation laws. Liang et al. [32] extended the parametrized MPP flux limiters to 2D scalar hyperbolic conservation laws.

Furthermore, the hyperbolic equations contain source terms, especially when the