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A Positivity-Preserving and Well-Balanced High Order Compact Finite Difference Scheme for Shallow Water Equations

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Abstract. We construct a positivity-preserving and well-balanced high order accurate finite difference scheme for solving shallow water equations under the fourth order compact finite difference framework. The source term is rewritten to balance the flux gradient in steady state solutions. Under a suitable CFL condition, the proposed compact difference scheme satisfies weak monotonicity, i.e., the average water height defined by the weighted average of a three-points stencil stays non-negative in forward Euler time discretization. Thus, a positivity-preserving limiter can be used to enforce the positivity of water height point values in a high order strong stability preserving Runge-Kutta method. A TVB limiter for compact finite difference scheme is also used to reduce numerical oscillations, without affecting well-balancedness and positivity. Numerical experiments verify that the proposed scheme is high-order accurate, positivity-preserving, well-balanced and free of numerical oscillations.

AMS subject classifications: 65M06, 65M12

Key words: Well-balanced, positivity-preserving, compact finite difference.

1 Introduction

The shallow water equations play an important role in the modeling and numerical simulation of flows in coastal water regions. The main goal of this paper is to con-

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struct high order accurate compact finite difference methods for the shallow water equations (SWEs), which are not only well-balanced for the still water steady state solutions but also positivity-preserving for water height. One difficulty for numerically solving shallow water equations system is to fulfill the *C-property* [1], which refers to keeping the stationary solution satisfying $h+b=constant$ and $hu=0$. For still water steady state solutions, flux gradients are nonzero but exactly balanced by the source term. There are many well-balanced schemes for shallow water equations in the literature, e.g., [4,8,22,24,26,27,33,34], including the weighted essentially non-oscillatory (WENO) schemes [25,32]. In the well-balanced scheme by using compatible discretization of slope source term over complex topography [19], and the scheme for a pre-balanced shallow water equation [28], water surface level rather than the conservative variable water height is solved as the reconstruction variable [41].

Another well-known difficulty is the appearance of wet and dry front for which negative water depth may emerge. Non-negativity or positivity of water height must be enforced to avoid non-physical phenomena and numerical instabilities. There are related positivity-preserving schemes in the literature, e.g., [2, 10, 13]. There are also schemes which are both well-balanced and positivity-preserving, e.g., [9,13,21,35,38–40].

In [36, 37], a positivity-preserving limiter for high order discontinuous Galerkin schemes is designed to enforce positivity without affecting the well-balancedness. The same approach in [36, 37] can also be used to construct positivity-preserving and well-balanced high order finite volume schemes. However, it is quite difficult to extend it to general finite difference schemes. In this paper, we will focus on constructing a positivity-preserving and well-balanced high order compact finite difference scheme. The compact finite difference scheme has good performance in terms of high resolution in the smooth region, but in discontinuity region even small oscillations can be distributed globally, which was analyzed in [7]. To solve shallow water equations with steady-state solution, a well-balanced WENO scheme using interpolation for variables for shallow water equation is proposed in [18]. A hybrid compact-WENO scheme is proposed in [42] where the WENO scheme is applied in discontinuous region while the compact finite difference scheme is used in smooth region. High order well-balanced weighted compact nonlinear schemes (WCNS) were proposed in [11].

In this paper, we design a positivity-preserving and well-balanced fourth order compact finite difference scheme for shallow water equations. Based on the weak monotonicity of fourth order compact finite difference scheme with forward Euler time discretization, a simple three-point stencil positivity-preserving limiter can be used to enforce the positivity of water height. Strong stability preserving (SSP) Runge-Kutta methods are used for the high order time discretization. To reduce oscillations, a total-variation-bounded (TVB) limiter is applied to the numerical flux computation [6]. However, the TVB limiter for the compact finite difference scheme [6] is significantly different from that for discontinuous Galerkin and finite volume schemes [5] because it is defined based on numerical flux. Due to this complication, it is not straightforward to use TVB limiter in a compact finite difference scheme without affecting positivity and well-balancedness. By