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The 2D Boussinesq-Navier-Stokes Equations with Logarithmically Supercritical Dissipation

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Abstract. We study the global well-posedness of the initial-value problem for the 2D Boussinesq-Navier-Stokes equations with dissipation given by an operator $\mathcal{L}$ that can be defined through both an integral kernel and a Fourier multiplier. When the operator $\mathcal{L}$ is represented by $\frac{|\xi|}{a(|\xi|)}$ with a satisfying $\lim_{|\xi|\to\infty}\frac{a(|\xi|)}{|\xi|^\sigma}=0$ for any $\sigma>0$, we obtain the global well-posedness. A special consequence is the global well-posedness of 2D Boussinesq-Navier-Stokes equations when the dissipation is logarithmically supercritical.

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1 Introduction

In this paper, we focused on the initial-value problem (IVP) for the Boussinseq-Navier-Stokes equations with dissipation given by a general integral operator,

$$\begin{cases} \partial_t u+u\cdot\nabla u+\mathcal{L}u=-\nabla p+\theta\mathbf{e}_2, \\ \partial_t\theta+u\cdot\nabla\theta=0, \\ \nabla\cdot u=0, \\ u(x,0)=u_0(x), \quad \theta(x,0)=\theta_0(x), \end{cases} \tag{1.1}$$

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where $u:\mathbb{R}^2\to\mathbb{R}^2$ is a vector field denoting the velocity, $\theta:\mathbb{R}^2\to\mathbb{R}$ is a scalar function, $\mathbf{e}_2$ is the unit vector in the x_2 direction, and $\mathcal{L}$ is a nonlocal dissipation operator defined by

$$\mathcal{L}f(x)=\text{ p.v. }\int_{\mathbb{R}^2}\frac{f(x)-f(y)}{|x-y|^2}m(|x-y|)dy \tag{1.2}$$

and $m:\ (0,\infty)\to(0,\infty)$ is a smooth, positive, non-increasing function, which obeys

(i) there exists $C_1>0$ such that

$$rm(r)\le C_1 \qquad \text{for all } r\le 1;$$

(ii) there exists $C_2>0$ such that

$$r|m'(r)|\le C_2 m(r) \qquad \text{for all } r>0;$$

(iii) there exists $\beta>0$ such that

$$r^\beta m(r) \text{ is non-increasing.}$$

This type of dissipation operator was introduced by Dabkowski, Kiselev, Silvestre and Vicol when they study the well-posedness of slightly supercritical active scalar equations [13]. As pointed out in [13], $\mathcal{L}$ can be equivalently defined by a Fourier multiplier, namely

$$\widehat{\mathcal{L}f}(\xi)=P(|\xi|)\widehat{f}(\xi) \tag{1.3}$$

for $P(|\xi|)=m(\frac{1}{|\xi|})$ when $P(\xi)$ satisfies the following conditions:

1. P satisfies the doubling condition: for any $\xi\in\mathbb{R}^2$,

$$P(2|\xi|)\le c_D P(|\xi|)$$

   with constant $c_D\ge 1$;

2. P satisfies the Hormander-Mikhlin condition (see [33]): for any $\xi\in\mathbb{R}^2$,

$$|\xi|^{|k|}|\partial_\xi^k P(|\xi|)|\le c_H P(|\xi|)$$

   for some constant $c_H\ge 1$, and for all multi-indices $k\in\mathbb{Z}^d$ with $|k|\le N$, with N only depending on c_D;

3. P has sub-quadratic growth at ∞, i.e.

$$\int_0^1 P(|\xi|^{-1})|\xi|d|\xi|<\infty$$