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SUPERCONVERGENCE AND *A POSTERIORI* ERROR ESTIMATES OF A LOCAL DISCONTINUOUS GALERKIN METHOD FOR THE FOURTH-ORDER INITIAL-BOUNDARY VALUE PROBLEMS ARISING IN BEAM THEORY

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Abstract. In this paper, we investigate the superconvergence properties and *a posteriori* error estimates of a local discontinuous Galerkin (LDG) method for solving the one-dimensional linear fourth-order initial-boundary value problems arising in study of transverse vibrations of beams. We present a local error analysis to show that the leading terms of the local spatial discretization errors for the k-degree LDG solution and its spatial derivatives are proportional to $(k+1)$-degree Radau polynomials. Thus, the k-degree LDG solution and its derivatives are $\mathcal{O}(h^{k+2})$ superconvergent at the roots of $(k+1)$-degree Radau polynomials. Computational results indicate that global superconvergence holds for LDG solutions. We discuss how to apply our superconvergence results to construct efficient and asymptotically exact *a posteriori* error estimates in regions where solutions are smooth. Finally, we present several numerical examples to validate the superconvergence results and the asymptotic exactness of our *a posteriori* error estimates under mesh refinement. Our results are valid for arbitrary regular meshes and for P^k polynomials with $k \geq 1$, and for various types of boundary conditions.

Key words. Local discontinuous Galerkin method; fourth-order initial-boundary value problems; Euler-Bernoulli beam equation; superconvergence; *a posteriori* error estimates.

1. Introduction

The goal of this paper is to investigate the superconvergence properties and develop a simple procedure to compute *a posteriori* error estimates of the spatial errors for the local discontinuous Galerkin (LDG) method applied to the following linear fourth-order initial-boundary value problem in one space dimension:

$$u_{tt} + u_{xxxx} = f(x,t), \quad x \in [0,L],\ t \in [0,T], \tag{1.1a}$$

subject to the initial conditions

$$u(x,0) = g(x), \quad u_t(x,0) = h(x), \quad x \in [0,L], \tag{1.1b}$$

and to one of the following four kinds of boundary conditions which are commonly encountered in practice ($t \in [0,T]$):

$$u(0,t) = u_1(t),\ u_{xx}(0,t) = u_2(t),\ u_x(L,t) = u_3(t),\ u_{xxx}(L,t) = u_4(t), \tag{1.1c}$$

$$u(0,t) = u_1(t),\ u_{xx}(0,t) = u_2(t),\ u(L,t) = u_3(t),\ u_{xx}(L,t) = u_4(t), \tag{1.1d}$$

$$u(0,t) = u_1(t),\ u_x(0,t) = u_2(t),\ u(L,t) = u_3(t),\ u_x(L,t) = u_4(t), \tag{1.1e}$$

$$u(0,t) = u(L,t),\ u_x(0,t) = u_x(L,t),\ u_{xx}(0,t) = u_{xx}(L,t),\ u_{xxx}(0,t) = u_{xxx}(L,t). \tag{1.1f}$$

In our analysis we assume that the interval $[0,T]$ is a finite time interval, and select the side conditions and the source, $f(x,t)$, such that the exact solution, $u(x,t)$, is

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a smooth function on $[0, L] \times [0, T]$. Even though the analysis in this paper is restricted to (1.1a), the same results can be directly generalized to the well-known Euler-Bernoulli beam equation with constant and variable geometrical and physical properties

$$(E(x)I(x)u_{xx})_{xx} + \rho(x)A(x)u_{tt} = f(x,t),$$

where $u(x, t)$ is the deflection of the neutral axis of the beam, $E(x)$ is the Young's modulus of elasticity, $I(x)$ is the area moment of inertia of the cross section with respect to its neutral midplane, $A(x)$ is the cross section in the yz-plane, $\rho(x)$ is the mass density per unit length, and $f(x, t)$ is the transverse load.

The fourth-order Euler-Bernoulli beam equation considered in this paper plays a very important role in both theory and applications. This is due to its use to describe a large number of physical and engineering phenomenons such as the flexural vibrations of a slender isotropic beam within the framework of Euler-Bernoulli assumptions. Several numerical schemes are proposed in the literature for solving (1.1a). Consult [11, 12, 14, 35, 36, 37, 41, 42, 47] and the references cited therein for more details. In this paper, we develop, analyze and test a superconvergent LDG method for solving (1.1). The proposed scheme is based on the fourth-order Runge-Kutta method approximation in time and on the LDG approximation in the spatial discretization. Our proposed scheme for solving the beam equation extends our previous work [16, 23] in which we investigated the convergence properties and the error estimates of the LDG method applied to the second-order wave and convection-diffusion equations in one space dimension.

The main motivation for the LDG method proposed in this paper originates from the LDG techniques which have been developed for convection-diffusion equations. The LDG finite element method considered here is an extension of the discontinuous Galerkin (DG) method aimed at solving ordinary and partial differential equations (PDEs) containing higher than first-order spatial derivatives. The DG method is a class of finite element methods using completely discontinuous piecewise polynomials for the numerical solution and the test functions. With discontinuous finite element bases, they capture discontinuities in, *e.g.*, hyperbolic systems with high accuracy and efficiency; simplify adaptive $h-$, $p-$, $r-$, refinements and produce efficient parallel solution procedures. The DG method was initially introduced by Reed and Hill in 1973 as a technique to solve neutron transport problems [44]. Lesaint and Raviart [40] presented the first numerical analysis of the method for a linear advection equation. Since then, DG methods have been used to solve ordinary and partial differential equations. Consult [32, 17] and the references cited therein for a detailed discussion of the history of DG method and a list of important citations on the DG method and its applications.

The LDG method for solving convection-diffusion problems was first introduced by Cockburn and Shu in [33]. They further studied the stability and error estimates for the LDG method. Castillo *et al.* [26] presented the first *a priori* error analysis for the LDG method for a model elliptic problem. They considered arbitrary meshes with hanging nodes and elements of various shapes and studied general numerical fluxes. They showed that, for smooth solutions, the L^2 errors in ∇u and in u are of order k and $k + 1/2$, respectively, when polynomials of total degree not exceeding k are used. Cockburn *et al.* [31] presented a superconvergence result for the LDG method for a model elliptic problem on Cartesian grids. They identified