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AN H^1-CONFORMING SOLENOIDAL BASIS FOR VELOCITY COMPUTATION ON POWELL-SABIN SPLITS FOR THE STOKES PROBLEM

JEFFREY M. CONNORS AND MICHAEL GAIEWSKI

Abstract. A solenoidal basis is constructed to compute velocities using a certain finite element method for the Stokes problem. The method is conforming, with piecewise linear velocity and piecewise constant pressure on the Powell-Sabin split of a triangulation. Inhomogeneous Dirichlet conditions are supported by constructing an interpolating operator into the solenoidal velocity space. The solenoidal basis reduces the problem size and eliminates the pressure variable from the linear system for the velocity. A basis of the pressure space is also constructed that can be used to compute the pressure after the velocity, if it is desired to compute the pressure. All basis functions have local support and lead to sparse linear systems. The basis construction is confirmed through rigorous analysis. Velocity and pressure system matrices are both symmetric, positive definite, which can be exploited to solve their corresponding linear systems. Significant efficiency gains over the usual saddle-point formulation are demonstrated computationally.

Key words. Divergence free, Powell-Sabin, Stokes, finite element, saddle-point.

1. Introduction

This paper relates to finite element computations for the incompressible Stokes problem in two dimensions. Given the real, simply connected and polygonal domain $\Omega \subset \mathbb{R}^2$, we consider the Dirichlet problem to solve for velocity $\mathbf{u} : \overline{\Omega} \to \mathbb{R}^2$ and pressure $p : \Omega \to \mathbb{R}$ such that

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{on } \Omega, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{on } \Omega, \tag{2}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega, \tag{3}$$

$$\text{and } \int_\Omega p \, d\mathbf{x} = 0, \tag{4}$$

where $\nu > 0$ is a constant viscosity parameter and $\mathbf{g}$ is a target set of boundary values for the velocity field. Bold font will be reserved for vectors and spaces of vector-valued functions.

There are countless research expositions related to the Stokes problem since it connects to many scientific models and problems in mathematics. Here, we are primarily interested in the computation of velocity and pressure variables using a certain finite element method where the incompressibility constraint (2) is ultimately satisfied pointwise over the domain, yielding a true solenoidal velocity field. In contrast, many methods only satisfy this condition in some weak (integrated) sense, and special mixed pairings of elements for the velocity and pressure are generally needed for strong incompressibility. A motivation is that solenoidal velocities break a coupling in the consistency error between velocity and pressure variables, so that velocity computation is not polluted by errors that should only affect the pressure

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accuracy. In the literature, this property is called *pressure robustness*. A review of methods may be found in [8].

These methods turn out to carry an additional potential benefit: to construct a locally-supported and solenoidal basis directly for the velocity space. Whereas a certain saddle-point problem is typically solved for the velocity and pressure variables, a solenoidal basis can be used to express the linear system in a block-triangular fashion that allows the velocity to be computed without computing the pressure. If the pressure is desired, it can then be calculated after via a separate solve, but in both cases the saddle-point solve is replaced by smaller, symmetric positive-definite systems. Few methods of this type exist at present, and it is not clear that a solenoidal basis can always have a local support. There are methods of discontinuous-Galerkin (DG) type [7, 10, 11]. Also, for Raviart-Thomas (RT), Brezzi-Douglass-Marini (BDM) and hybridized locally-DG mixed elements that possess a weak divergence but not a weak gradient, see [2, 15, 1, 16]. In the H^1-conforming case there is method with fourth-order polynomial velocities [12] and a method with first-order velocities [13].

The purpose of this paper is to develop the decoupled velocity and pressure computations for the mixed pair in [4]. The method enforces solenoidal, first-order polynomial velocities and is H^1-conforming. The element order is the same as the method in [13], but the finite element mesh is quite different. We use Powell-Sabin splits that subdivide triangles of a fairly general mesh into six subtriangles (detailed later), whereas the meshes of [13] use rectangular meshes and subdivide each rectangle into four triangles. This latter meshing approach may be less convenient for some applications. Besides the geometry, the method of this paper allows for a local macro-element assembly requiring only the six local triangles of the Powell-Sabin split at once, grouping nodal basis functions for four nodes. The method of [13] refers to macro-element basis functions over a nine-rectangle grid of four triangles per rectangle, hence thirty-six triangles, and ultimately groups thirteen nodal functions together.

We summarize the paper contents as follows. Details of the Powell-Sabin mesh and finite element method for Stokes are given in Section 2, along with some preliminary technical lemmas. This includes a discussion about handling Dirichlet boundary conditions. In Section 3 we construct a solenoidal basis for the finite element velocity space with local support that can accommodate Dirichlet conditions. The properties are proved, and we include details of the local construction for implementation. In Section 4 we construct a local basis of the pressure space such that each basis function is the divergence of a known basis function in the (non-solenoidal) discrete velocity space, with proof. This pressure basis requires a subset of vertices to be marked using a graph-theoretic *spanning tree*, for which purpose there is already an inexpensive algorithm due to Kruskal [9]. The pressure basis is fairly simple to implement once this is done. Computational examples are given in Section 5 comparing the velocity and pressure computations to the classical saddle-point system, the latter using the usual non-solenoidal basis for velocity. A summary discussion is provided in Section 6.

2. A finite element method on Powell-Sabin splits

This section provides some mathematical preliminaries and then outlines the finite element spaces studied, focusing on a specific instance from amongst those discussed in [4].