Journal of Computational Mathematics
Vol.42, No.3, 2024, 617–637.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2301-m2022-0099

SEMI-PROXIMAL POINT METHOD FOR NONSMOOTH CONVEX-CONCAVE MINIMAX OPTIMIZATION*

Yuhong Dai
LSEC, ICMSEC, AMSS, Chinese Academy of Sciences, Beijing 100190, China
School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
Email: dyh@lsec.cc.ac.cn
Jiani Wang[1)]
Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
Email: wjiani@lsec.cc.ac.cn
Liwei Zhang
School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China
Email: lwzhang@dlut.edu.cn

Abstract

Minimax optimization problems are an important class of optimization problems arising from modern machine learning and traditional research areas. While there have been many numerical algorithms for solving smooth convex-concave minimax problems, numerical algorithms for nonsmooth convex-concave minimax problems are rare. This paper aims to develop an efficient numerical algorithm for a structured nonsmooth convex-concave minimax problem. A semi-proximal point method (SPP) is proposed, in which a quadratic convex-concave function is adopted for approximating the smooth part of the objective function and semi-proximal terms are added in each subproblem. This construction enables the subproblems at each iteration are solvable and even easily solved when the semiproximal terms are cleverly chosen. We prove the global convergence of our algorithm under mild assumptions, without requiring strong convexity-concavity condition. Under the locally metrical subregularity of the solution mapping, we prove that our algorithm has the linear rate of convergence. Preliminary numerical results are reported to verify the efficiency of our algorithm.

Mathematics subject classification: 90C30.
Key words: Minimax optimization, Convexity-concavity, Global convergence, Rate of convergence, Locally metrical subregularity.

1. Problem Setting

In this paper, we consider the following nonsmooth minimax optimization problem:

$$\min_{x\in\mathcal{X}}\max_{y\in\mathcal{Y}} L(x,y) := f(x) + K(x,y) - g(y), \tag{1.1}$$

where $K : \mathcal{X}\times\mathcal{Y}\to\Re$ is a continuously differentiable convex-concave function, and $f:\mathcal{X}\to\overline{\Re}$, $g:\mathcal{Y}\to\overline{\Re}$ are proper lower semi-continuous convex functions. $\mathcal{X}$ and $\mathcal{Y}$ be two finite-dimensional real Hilbert spaces equipped with a scalar product $\langle\cdot,\cdot\rangle$ and its induced norm $\|\cdot\|$.

* Received April 20, 2022 / Revised version received July 26, 2022 / Accepted January 3, 2023 / Published online April 7, 2023 /
1) Corresponding author

The mathematical model (1.1) covers a lot of interesting convex-concave minimax problems appeared in the literature. We only list two examples here.

Example 1.1. Let $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$ be two closed convex sets. The following constrained minimax optimization problem are frequently studied in the literature:

$$\min_{x \in X} \max_{y \in Y} K(x, y). \tag{1.2}$$

Obviously, the constrained minimax optimization problem (1.2) can be written as the form of problem (1.1) if we set $f(x) = \delta_X(x)$ and $g(y) = \delta_Y(y)$.

Example 1.2. In Example 1.1, let $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$ be two closed convex sets specified as

$$X = \{x \in \mathcal{X} : G(x) \in C\}, \quad Y = \{y \in \mathcal{Y} : H(y) \in D\},$$

where C and D are closed convex sets in some finite dimensional spaces and the set-valued mappings

$$x : \rightarrow \ G(x) - C, \quad y : \rightarrow \ H(y) - D$$

are graph-convex (under this assumption X and Y are convex sets). Then the constrained minimax optimization problem (1.2) can be written as the form of problem (1.1) if we set $f(x) = \delta_C(G(x))$ and $g(y) = \delta_D(H(y))$.

The study of algorithms for solving convex-concave minimax problems of the form (1.1) is active. For the case when K is a bilinear function, there are many publications about constructing and analyzing numerical algorithms for the minimax problem. The first work was due to Arrow *et al.* [1], where they proposed an alternating coordinate method, leaving the convergence unsolved. Nemirovski [14] considered the following minimax problem:

$$\min_{x} \max_{y \in Y} g(x) + x^T A y + h^T y,$$

where Y is a compact convex set and g is a $\mathcal{C}^{1,1}$ convex function. He proposed a mirror-prox algorithm which returns an approximate saddle point within the complexity of $\mathcal{O}(1/\varepsilon)$. Nesterov [15] developed a dual extrapolation algorithm for solving variational inequalities which owns the complexity bound $\mathcal{O}(1/\varepsilon)$ for Lipschitz continuous operators and applied the algorithm to bilinear matrix games. Chen *et al.* [7] presented a novel accelerated primal-dual (APD) method for solving this class of minimax problems, and showed that the APD method achieves the same optimal rate of convergence as Nesterov's smoothing technique. Chambolle and Pock [4] proposed a first-order primal-dual algorithm and established the convergence of the algorithm. Later, Chambolle and Pock provided the ergodic convergence rate [5] and explored the rate of convergence for accelerated primal-dual algorithms [6].

For smooth convex-concave minimax problems when K is not bilinear, many numerical algorithms are proposed such as the projection method [19], extragradient method [10], Tseng's accelerated proximal gradient algorithm [21], catalyst algorithm framework [24]. Recently, Mokhtari *et al.* [12] proposed algorithms admitting a unified analysis as approximations of the classical proximal point method for solving saddle point problems. Mokhtari *et al.* [13] proved that the optimistic gradient and extra-gradient methods achieve a convergence rate of $\mathcal{O}(1/k)$ for smooth convex-concave saddle point problems. Yoon and Ryu [25] combined extra-gradient