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ONE-PARAMETER FINITE DIFFERENCE METHODS AND THEIR ACCELERATED SCHEMES FOR SPACE-FRACTIONAL SINE-GORDON EQUATIONS WITH DISTRIBUTED DELAY*

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Abstract

This paper deals with numerical methods for solving one-dimensional (1D) and two-dimensional (2D) initial-boundary value problems (IBVPs) of space-fractional sine-Gordon equations (SGEs) with distributed delay. For 1D problems, we construct a kind of one-parameter finite difference (OPFD) method. It is shown that, under a suitable condition, the proposed method is convergent with second order accuracy both in time and space. In implementation, the preconditioned conjugate gradient (PCG) method with the Strang circulant preconditioner is carried out to improve the computational efficiency of the OPFD method. For 2D problems, we develop another kind of OPFD method. For such a method, two classes of accelerated schemes are suggested, one is alternative direction implicit (ADI) scheme and the other is ADI-PCG scheme. In particular, we prove that ADI scheme can arrive at second-order accuracy in time and space. With some numerical experiments, the computational effectiveness and accuracy of the methods are further verified. Moreover, for the suggested methods, a numerical comparison in computational efficiency is presented.

Mathematics subject classification: 65M06, 65M12.
Key words: Fractional sine-Gordon equation with distributed delay, One-parameter finite difference methods, Convergence analysis, ADI scheme, PCG method.

1. Introduction

The sine-Gordon equations (SGEs) are a kind of important partial differential equations used to model some practical problems in electrodynamics [1–3], nonlinear optics [4], particle physics [5] and the other related scientific fields. In order to give a detailed description to the physical background of SGEs, as an example, we recall the following SGE used to characterize the long Josephson junction in an electrodynamic system ([2]):

$$\frac{\partial^2}{\partial t^2}u(x,t)=\kappa\frac{\partial^2}{\partial x^2}u(x,t)+\beta\sin(u(x,t)), \tag{1.1}$$

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where $u(x,t)$ denotes the superconducting phase difference across the Josephson junction and $\kappa > 0$ and β are the two real parameters of the system. Due to the existence of quasi-particle tunnel current and external bias current, dissipative term $\sigma\frac{\partial}{\partial t}u(x,t)$ ($\sigma \geq 0$) and source term $f(x,t)$ were introduced and thus an extended SGE was derived as follows ([1]):

$$\frac{\partial^2}{\partial t^2}u(x,t) + \sigma\frac{\partial}{\partial t}u(x,t) = \kappa\frac{\partial^2}{\partial x^2}u(x,t) + \beta\sin(u(x,t)) + f(x,t). \tag{1.2}$$

In view of the fact that time-dependent problems generally have the aftereffect phenomenon, some researchers introduced distributed delay $\int_{t-s}^{t} e^{-\alpha(t-\eta)}\frac{\partial}{\partial\eta}u(x,\eta)d\eta$ ($\alpha, s > 0$) into the above equation to produce the extended SGE ([6])

$$\begin{aligned}&\frac{\partial^2}{\partial t^2}u(x,t) + \sigma\frac{\partial}{\partial t}u(x,t)\\ &= \kappa\frac{\partial^2}{\partial x^2}u(x,t) - \int_{t-s}^{t} e^{-\alpha(t-\eta)}\frac{\partial}{\partial\eta}u(x,\eta)d\eta + \beta\sin(u(x,t)) + f(x,t).\end{aligned} \tag{1.3}$$

On the other hand, in order to model the nonlocal Josephson junction, by replacing the term $\frac{\partial^2}{\partial x^2}u(x,t)$ in (1.2) with a nonlocal operator

$$\mathcal{H}(u_x(x,t)) := \frac{1}{\pi}v.p.\int_{-\infty}^{+\infty}\frac{1}{\hat{x}-x}\frac{\partial u(\hat{x},t)}{\partial\hat{x}}d\hat{x},$$

the following nonlocal SGE was presented ([7]):

$$\frac{\partial^2}{\partial t^2}u(x,t) + \sigma\frac{\partial}{\partial t}u(x,t) = \kappa\mathcal{H}(u_x(x,t)) + \beta\sin(u(x,t)) + f(x,t). \tag{1.4}$$

In Ray [9], the author pointed out that $\frac{\partial^2}{\partial x^2}u(x,t)$ and $\mathcal{H}(u_x(x,t))$ are just the special cases when the order γ ($1 < \gamma \leq 2$) of the Riesz space-fractional derivative tends to 2 and 1, respectively, where the γ-order Riesz fractional derivative $\frac{\partial^\gamma}{\partial|x|^\gamma}u(x,t)$ is defined by

$$\begin{aligned}\frac{\partial^\gamma}{\partial|x|^\gamma}u(x,t) = &-\frac{1}{2\cos(\frac{\gamma\pi}{2})\Gamma(2-\gamma)}\frac{\partial^2}{\partial x^2}\\ &\times\left(\int_a^x\frac{u(\zeta,t)}{(x-\zeta)^{\gamma-1}}d\zeta + \int_x^b\frac{u(\zeta,t)}{(\zeta-x)^{\gamma-1}}d\zeta\right), \quad a \leq x \leq b,\end{aligned} \tag{1.5}$$

in which $\Gamma(\cdot)$ is the Gamma function. Based on this finding, Eqs. (1.2) and (1.4) were further generalized into the following space-fractional SGE ([8–12]):

$$\frac{\partial^2}{\partial t^2}u(x,t) + \sigma\frac{\partial}{\partial t}u(x,t) = \kappa\frac{\partial^\gamma}{\partial|x|^\gamma}u(x,t) + \beta\sin(u(x,t)) + f(x,t), \quad 1 < \gamma \leq 2. \tag{1.6}$$

Similar to Eq. (1.3), we consider the aftereffect phenomenon and introduce distributed delay $\int_{t-s}^{t} e^{-\alpha(t-\eta)}\frac{\partial}{\partial\eta}u(x,\eta)d\eta$ into Eq. (1.6). This generates the following space-fractional SGE with distributed delay:

$$\begin{aligned}&\frac{\partial^2}{\partial t^2}u(x,t) + \sigma\frac{\partial}{\partial t}u(x,t)\\ &= \kappa\frac{\partial^\gamma}{\partial|x|^\gamma}u(x,t) - \int_{t-s}^{t} e^{-\alpha(t-\eta)}\frac{\partial}{\partial\eta}u(x,\eta)d\eta + \beta\sin(u(x,t)) + f(x,t).\end{aligned} \tag{1.7}$$