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# SOLVING SYSTEMS OF PHASELESS EQUATIONS VIA RIEMANNIAN OPTIMIZATION WITH OPTIMAL SAMPLING COMPLEXITY*

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**Abstract**

A Riemannian gradient descent algorithm and a truncated variant are presented to solve systems of phaseless equations $|\boldsymbol{Ax}|^2 = \boldsymbol{y}$. The algorithms are developed by exploiting the inherent low rank structure of the problem based on the embedded manifold of rank-1 positive semidefinite matrices. Theoretical recovery guarantee has been established for the truncated variant, showing that the algorithm is able to achieve successful recovery when the number of equations is proportional to the number of unknowns. Two key ingredients in the analysis are the restricted well conditioned property and the restricted weak correlation property of the associated truncated linear operator. Empirical evaluations show that our algorithms are competitive with other state-of-the-art first order nonconvex approaches with provable guarantees.

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*Key words:* Phaseless equations, Riemannian gradient descent, Manifold of rank-1 and positive semidefinite matrices, Optimal sampling complexity.

## 1. Introduction

In this paper we are interested in finding a vector $\boldsymbol{x} \in \mathbb{R}^n$ or $\mathbb{C}^n$ which solves the following system of phaseless equations:

$$|\boldsymbol{Ax}|^2 = \boldsymbol{y}, \tag{1.1}$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}/\mathbb{C}^{m\times n}$ and $\boldsymbol{y} \in \mathbb{R}^m$ are both known. Compared with linear systems of the form $\boldsymbol{Ax} = \sqrt{\boldsymbol{y}}$, it is self-evident that after taking the entrywise modulus phase information is missing from (1.1). Thus, seeking a solution to (1.1) is often referred to as generalized phase retrieval, extending the classical phase retrieval problem where $\boldsymbol{A}$ is a Fourier type matrix to a general setting. Phase retrieval arises in a wide range of practical context such as X-ray crystallography [1], diffraction imaging [2] and microscopy [3], where it is hard or infeasible to record the phase information when detecting an object. Many heuristic yet effective algorithms have been developed for phase retrieval, for example, Error Reduction, Hybrid Input Output and other variants [4–7].

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Injectivity has been investigated in [8,9], showing that $m \geq 2n-1$ real generic measurements or $m \geq 4n-4$ complex generic measurements are sufficient to determine a unique solution of (1.1) up to a global phase vector. Despite this, solving systems of phaseless equations is computationally intractable. For simplicity, let us consider the real case. Then one can immediately see the combinatorial nature of the problem since there are $2^m$ possible signs for $\boldsymbol{y}$. In fact, a very simple instance of (1.1) can be reduced to the NP-hard stone problem [10].

Over the past few years computing methods with provable guarantees have received extensive investigations for solving systems of phaseless equations, typically based on the Gaussian measurement model. In a pioneering work by Candès *et al.* [11], a convex relaxation via trace norm minimization, known as PhaseLift, was studied. The approach was developed based on the fact that (1.1) can be cast as a rank-1 positive semidefinite matrix recovery problem. Inspired by the work on low rank matrix recovery, it was established that under the Gaussian measurement model PhaseLift was able to find the solution of (1.1) with high probability provided that[1] $m \gtrsim n \log n$. This sampling complexity was subsequently sharpened to $m \gtrsim n$ in [12]. The recovery guarantee of PhaseLift under coded diffraction model was studied in [13, 14]. There were also several other convex relaxation methods for solving systems of phaseless equations, see for example [15–18].

Convex methods are amenable to detailed analysis, but they are not computationally desirable for large scale problems. Thus more scalable yet still provable nonconvex methods have received particular attention recently. A resampled variant of Error Reduction has been investigated in [19], showing that $m \gtrsim n \log^3 n + n \log n \log(1/\epsilon) \log\log(1/\epsilon)$ number of measurements are sufficient for the algorithm to attain an $\epsilon$-accuracy. A gradient descent algorithm called Wirtinger Flow (WF) was developed based on an intensity-based loss function [20], and it was shown that the algorithm could achieve successful recovery provided $m \gtrsim n \log n$ [20, 21]. A variant of WF, known as Truncated Wirtinger Flow (TWF), was introduced in [10] based on the Poisson loss function, which could achieve successful recovery under the optimal sampling complexity $m \gtrsim n$. In [22], the algorithm was analyzed when median truncation was used. Another gradient descent algorithm, termed Truncated Amplitude Flow (TAF), was developed in [23] based on an amplitude-based loss function. Optimal theoretical recovery guarantee of TAF was similarly established under the Gaussian measurement model. In [24], the classical Kaczmarz method for solving systems of linear equations was extended to solve the generalized phase retrieval problem. The optimal sampling complexity for the successful recovery of the Kaczmarz method was established in [25, 26] when the unknown vector and the measurement matrix were both real.

The nonconvex algorithms mentioned in the last paragraph are all analyzed based on some local geometry and hence closeness of the initial guess to the ground truth is required [10, 20, 23, 25, 26]. In contrast, there is a line of research which attempts to study the global geometry of related problems, see [27, 27, 28] and references therein. Many algorithms have also been designed to utilize the global geometry effectively [29–32]. We omit further details as it is beyond the scope of this paper and interested readers are referred to the references.

**Main contributions.** In this paper we propose a Riemannian gradient descent algorithm for solving systems of phaseless equations. The algorithm is developed by exploiting the inherent low rank structure of (1.1) based on the embedded manifold of low rank matrices, similar to the Riemannian gradient descent algorithm for the low rank matrix recovery problem studied

[1] The notation $m \gtrsim f(n)$ means that there exists an absolute constant $C > 0$ such that $m \geq C \cdot f(n)$.