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A NEW GLOBAL OPTIMIZATION ALGORITHM FOR MIXED-INTEGER QUADRATICALLY CONSTRAINED QUADRATIC FRACTIONAL PROGRAMMING PROBLEM*

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Abstract

The mixed-integer quadratically constrained quadratic fractional programming (MIQCQFP) problem often appears in various fields such as engineering practice, management science and network communication. However, most of the solutions to such problems are often designed for their unique circumstances. This paper puts forward a new global optimization algorithm for solving the problem MIQCQFP. We first convert the MIQCQFP into an equivalent generalized bilinear fractional programming (EIGBFP) problem with integer variables. Secondly, we linearly underestimate and linearly overestimate the quadratic functions in the numerator and the denominator respectively, and then give a linear fractional relaxation technique for EIGBFP on the basis of non-negative numerator. After that, combining rectangular adjustment-segmentation technique and midpoint-sampling strategy with the branch-and-bound procedure, an efficient algorithm for solving MIQCQFP globally is proposed. Finally, a series of test problems are given to illustrate the effectiveness, feasibility and other performance of this algorithm.

Mathematics subject classification: 90C57, 90C26.
Key words: Global optimization, Branch and bound, Quadratic fractional programming, Mixed integer programming.

1. Introduction

Consider the following class of mixed integer quadratically constrained quadratic fractional programming (MIQCQFP) problems:

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$$
\text{(MIQCQFP)} \quad \begin{cases} \min f(x) = \dfrac{f_0(x)}{g(x)}, & \\ \text{s.t. } f_i(x) \le 0, & i = 1, 2, \ldots, N, \\ x \in X = \{x \in \mathbb{R}^n \mid Ax \le b\}, & \\ x_j \in \mathbb{Z}, & j \in N_I \subseteq \{1, 2, \ldots, n\}, \end{cases}
$$

where $f_i(x) = x^T Q_i x + c_i^T x + d_i, i = 0, 1, 2, \ldots, N$ with

$$
\begin{aligned} &Q_i \in \mathbb{R}^{n \times n}, && i = 0, 1, 2, \ldots, N, \\ &c_i \in \mathbb{R}^n, && i = 0, 1, 2, \ldots, N, \\ &d_i \in \mathbb{R}, && i = 0, 1, 2, \ldots, N, \end{aligned}
$$

$g(x) = x^T Hx + h^T x + q$ with $H \in \mathbb{R}^{n \times n}$, $h \in \mathbb{R}^n$, $q \in \mathbb{R}$ and X is a nonempty bounded set with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$; T denotes the transpose of a vector (or matrix) (for example, c_i^T is the transpose of a vector c_i); there is no convex (or concave) assumption for all functions; note that all forms of inequality constraints can contain all forms of constraints, because any equality constraints can be replaced by two inequality constraints; $\mathbb{Z}$ denotes a set of all integers; N_I is an integer index set. There will always be $g(x) > 0$ or $g(x) < 0$ for the denominator $g(x)$ in the objective function according to its continuity and nonzero property. If $g(x) < 0$ for some $x \in X$, we can make

$$
\frac{f_0(x)}{g(x)} = \frac{-f_0(x)}{-g(x)}.
$$

Then the original problem keeps unaltered, and the denominator becomes positive. In addition, if there is an $x \in X$ such that the numerator $f_0(x) < 0$, we can construct $f_0(x) + Mg(x) \ge 0$ with a sufficiently large positive number M, then

$$
\frac{f_0(x)}{g(x)} = \frac{f_0(x) + Mg(x)}{g(x)} - M,
$$

obviously, these two problems

$$
\min_{x \in X} \frac{f_0(x)}{g(x)} \quad \text{and} \quad \min_{x \in X} \frac{f_0(x) + Mg(x)}{g(x)}
$$

share the same solution. Therefore, throughout this paper, without loss of generality, we assume $f_0(x) \ge 0$ for any $x \in X$.

Problem MIQCQFP and its special cases are ubiquitous in the real world. From a computational point of view, the MIQCQFP may be difficult to solve because it has three kinds of non-convexity, one is the possible integer variable, one is the non-convex quadratic term in the objective function and/or constraint function, and the other is the fractional form of the objective function. Numerous variants of this problem are also divided into several categories according to the form of their objective functions or constraints. For example, when the constraints of MIQCQFP are only linear, Bomze and Amaral [3] pointed out that such problems arise from many application problems, such as optimizing communication or social networks, or studying game theory problems caused by genetics; they include several APX-hard subclasses: the maximum cut problem, k-densest subgraph problem and several variants thereof, or ternary fractional quadratic optimization problem (TFQP); also, they add rich evidence of common positive optimization methods and reveal possible new approximation strategies combining continuous optimization and discrete optimization techniques in the field of (fractional)