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Towards the Efficient Calculation of Quantity of Interest from Steady Euler Equations I: A Dual-Consistent DWR-Based *h*-Adaptive Newton-GMG Solver

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Abstract. The dual consistency is an important issue in developing stable DWR error estimation towards the goal-oriented mesh adaptivity. In this paper, such an issue is studied in depth based on a Newton-GMG framework for the steady Euler equations. Theoretically, the numerical framework is redescribed using the Petrov-Galerkin scheme, based on which the dual consistency is depicted. It is found that for a problem with general configuration, a boundary modification technique is an effective approach to preserve the dual consistency in our numerical framework. Numerically, a geometrical multigrid is proposed for solving the dual problem, and a regularization term is designed to guarantee the convergence of the iteration. The following features of our method can be observed from numerical experiments, i). a stable numerical convergence of the quantity of interest can be obtained smoothly for problems with different configurations, and ii). towards accurate calculation of quantity of interest, mesh grids can be saved significantly using the proposed dual-consistent DWR method, compared with the dual-inconsistent one.

AMS subject classifications: 65N08, 65N22, 65N50

Key words: Newton-GMG, DWR-based adaptation, finite volume method, dual consistency, *h*-adaptivity, steady Euler equations.

1 Introduction

Accurate calculation of quantity of interest is important in applications such as optimal design of vehicle's shape. However, obtaining a precise value of quantity of interest is

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time-consuming usually. Then mesh adaptation is a suitable strategy to figure out this issue. Pointed by [25], mesh generation and adaptivity continue to be significant bottlenecks in the computational fluid dynamics workflow. In order to obtain an economical mesh, developing a good criterion for adaptation is of major concern.

To accurately calculate the quantity of interest, dual-weighted residual-based mesh adaptation is a popular approach. There have been many mature methods in the development of the DWR-based h-adaptivity. As pointed out by [5], adjoint-based techniques have become more widely used in recent years as they have more solid theoretical foundations for error estimation and control analysis. From these theoretical analyses, it is realized that the dual consistency property is of vital importance to guarantee error estimation as well as convergence behavior. In [20], dual consistency is firstly analyzed, showing that the dual-weighted residual framework based on a dual-inconsistent mesh adaptation may lead to dual solutions with oscillations. Nevertheless, the dual consistency is not usually satisfied under certain issues, which results from the mismatch between the quantity of interest and the boundary conditions. In order to address this issue, the modification to the numerical schemes shall be realized. Then, Hartmann [9] proposed a series of boundary modification techniques to extend the analysis to general configurations.

In order to derive a dual-consistent framework, compatibility and consistency should be discussed accordingly. Firstly, compatibility puts emphasis on the dual part, whereby the quantity of interest generated by the functional of dual solutions should theoretically be equal to that of the primal solutions. Giles utilized a matrix representation approach to derive a continuous version of dual equations of Euler equations with different kinds of boundary conditions [6,23]. However, under certain boundary conditions, dual equations should be equipped with non-trivial strong boundary conditions to ensure the dual consistency. To facilitate the implementation of the DWR method in a discretized finite volume scheme, Darmofal [26,27] developed a fully discrete method, which is easier to implement. Secondly, the consistency part highlights the importance of the discretization method. Analysis of the discontinuous Galerkin scheme by [8,20] has shown that dual consistency requires specific prerequisites to be met in the discretization process. [10] identified the summation by parts as a significant factor in contributing to the dual inconsistency issue, thereby underscoring the indispensable role of discretization in the computational scheme. This finding puts emphasis on the necessity for improved accuracy in discretization methods for numerical solutions. Moreover, to apply the method for hp-adaptation, Vít Dolejŏí et al. have developed algorithms for solving adjoint-based problems [3,24], which have been applied to the Euler equations in [4].

In our previous work, focusing on the steady Euler equations, a competitive Newton-GMG solver has been developed for an efficient solution [11,12,17,18,29]. Furthermore, the DWR implementation has been discussed in [13,14,21]. However, the important issue, dual consistency, has not been considered in this framework. While the dual-weighted residual method was previously utilized for adaptation in our studies, it should be noted that failure to account for dual consistency will result in unforeseen phenomena. For